$\angle XA^*B^* = \pi - \angle XAB$, $\angle XB^*A^* = \pi - \angle XBA$, we have $\angle AXB + \angle BAX + \angle ABX = \angle ACB + \angle BAC + \angle ABC$. Hence, by Girard's formula, we have $|ABX| = |ABC|$. ■

For a different proof of Lexell's theorem, see L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer-Verlag, Berlin, 1972, p. 23.

College of Education, Ryukyu University, Nishihara, Okinawa, 903-0213 Japan.
hmaehara@edu.u-ryukyu.ac.jp

A Characteristic Property of Differentiation

Khristo Boyadzhiev

We offer here a simple exercise in calculus with a flavor of functional analysis. The differentiation operator $D: f \to f'$ is a fundamental operator in calculus and it is interesting to consider what properties distinguish it from all other operators on functions. One important theorem says that if a differentiable function $f(x)$ has a relative minimum (or maximum) at $x = a$, then $f'(a) = 0$. As we shall see now, this property "almost" characterizes D.

Notation. For convenience we consider only polynomials. Let P be the set of all polynomials and let p_n, $n = 0, 1, \ldots,$ be the basic polynomials:

$$p_0(x) = 1, p_1(x) = x, \ldots, p_n(x) = x^n, \ldots .$$

When $\delta : P \to P$ is a linear operator, we denote its action on $p \in P$ by $\delta[p]$. Thus $\delta[p]$ is again a polynomial and its value at some number x is written as $\delta[p](x)$.

Theorem 1. *Let $\delta : P \to P$ be a linear operator. Then the following are equivalent:*

(i) *If p has a relative minimum at $x = a$, then $\delta[p](a) = 0$.*
(ii) $\delta = \delta[p_1]D$.

In particular, if $\delta[p_1] = p_0$, then $\delta = D$. (Here "minimum" can be replaced by maximum.")

Proof: The implication (ii) → (i) is immediate, so we focus on (i) → (ii). First we want to show that every linear operator on P has a convenient general form. By Taylor's formula, for any polynomial p and any number a:

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(a)}{k!}(x-a)^k.$$

The sum is finite and we write "∞" just for convenience. Applying δ to both sides (as polynomials of x, with a fixed) we obtain

$$\delta[p] = \sum_{k=0}^{\infty} \frac{p^{(k)}(a)}{k!}\delta\left[(x-a)^k\right] \tag{1}$$

where $\delta[(x-a)^k]$ are polynomials of x. Setting

$$g_k(a) = \frac{1}{k!}\delta\left[(x-a)^k\right](a), \tag{2}$$

i.e., evaluating these polynomials at $x = a$, we define new polynomials g_k, $k = 0, 1, \dots$.

It is clear that (2) defines functions of the variable a, but why are these functions polynomials? Good question! To answer it we use the binomial formula for $(x-a)^k$ and the linearity of δ. Now (1) can be written in the form

$$\delta[p](a) = \sum_{k=0}^{\infty} g_k(a) p^{(k)}(a)$$

or simply

$$\delta[p] = g_0 + g_1 p' + g_2 p'' + \cdots . \tag{3}$$

This is the representation we need: the action of δ is expressed in a simple way in terms of the polynomials g_k. Notice that we did not use here property (i). Therefore, the general representation (3) is true for every linear operator on the polynomials.

It turns out that under condition (i) we have $g_k = 0$ for every $k > 1$. Indeed, consider the polynomial

$$f(x) = \frac{1}{2}(x-a)^2 + \frac{\lambda}{k!}(x-a)^k$$

which is specially designed to serve our purpose. Here a and λ are arbitrary real numbers and the integer $k > 2$. We have $f'(a) = 0$, $f''(a) = 1$, so f has a relative minimum at $x = a$. According to property (i)

$$\delta[f](a) = g_2(a) + \lambda g_k(a) = 0$$

Since this is true for all λ and a, we conclude that $g_k = 0$ identically for all $k > 1$. Also, $g_0 = \delta[p_0] = 0$, as p_0 has minimum at each number. Finally, using the linearity of δ we obtain

$$\begin{aligned} g_1(a) &= \delta[(x-a)](a) = \delta[p_1 - a p_0](a) \\ &= \delta[p_1](a) - a\delta[p_0](a) = \delta[p_1](a). \end{aligned}$$

Therefore, the representation (3) turns into

$$\delta[p] = \delta[p_1]p'$$

for every polynomial p. ■

The same proof gives the following.

Theorem 2. *Let $\delta : P \to P$ be a linear operator with the following property (Minimum Principle): $\delta[p](a) \geq 0$ whenever a polynomial p has a relative minimum at some number $x = a$. Then $\delta[p] = g_1 p' + g_2 p''$ for all $p \in P$, where the polynomials g_1, g_2 are defined in* (2) *and $g_2 \geq 0$.*

Theorem 2 naturally extends to polynomials of many variables: any linear operator $\delta : P \to P$ satisfying the Minimum Principle is a second-order elliptic partial differential operator. For instance, Markov processes (semigroups) in diffusion theory have generators that satisfy the Minimum Principle **[1]**. Therefore, we conclude that diffusion in nature is governed by second-order elliptic partial differential operators. Theorem 2 (in a different form) originates from A. Kolmogorov. Some others contributed to it, providing modifications and extensions: comments and references can be found in [**1**, Chapter 5] and [**2**, Chapter XIII].

REFERENCES

1. E. B. Dynkin, *Markov Processes*, vol 1., Springer-Verlag, Berlin, 1965.
2. K. Yosida, *Functional Analysis*, Springer-Verlag, New York, 1980.

Ohio Northern University, Ada, Ohio 45810.
k-boyadzhiev@onu.edu

A Weighted Mixed-Mean Inequality

Kiran S. Kedlaya

In [**4**], the author established the following inequality conjectured by Holland [**3**]. Unbeknownst to either of these parties, the same inequality had been earlier announced by Nanjundiah [**8**] without proof.

Theorem 1. *Let $x_1, x_2, \dots, x_n$ be positive real numbers. The arithmetic mean of the numbers*

$$x_1, \sqrt{x_1 x_2}, \dots, \sqrt[n]{x_1 x_2 \cdots x_n}$$

does not exceed the geometric mean of the numbers

$$x_1, \frac{x_1 + x_2}{2}, \dots, \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Here we prove the following weighted extension of Theorem 1.

Theorem 2. *Let $x_1, \dots, x_n, w_1, \dots, w_n$ be positive real numbers, and define $s_i = w_1 + \cdots + w_i$ for $i = 1, \dots, n$. Assume that*

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \cdots \geq \frac{w_n}{s_n}. \tag{1}$$

Then the weighted arithmetic mean of the numbers

$$x_1, x_1^{w_1/s_2} x_2^{w_2/s_2}, \dots, x_1^{w_1/s_n} x_2^{w_2/s_n} \cdots x_n^{w_n/s_n}$$

does not exceed the weighted geometric mean of the numbers

$$x_1, \frac{w_1}{s_2} x_1 + \frac{w_2}{s_2} x_2, \dots, \frac{w_1}{s_n} x_1 + \frac{w_2}{s_n} x_2 + \cdots + \frac{w_n}{s_n} x_n$$

when each mean is taken with weights $w_1/s_n, w_2/s_n, \dots, w_n/s_n$. In other words,

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{i} \frac{w_j}{s_i} x_j \right)^{w_i/s_n} \geq \sum_{j=1}^{n} \frac{w_j}{s_n} \prod_{i=1}^{j} x_i^{w_i/s_j}. \tag{2}$$