The Reciprocity Law for Dedekind Sums via the Constant Ehrhart Coefficient

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1. Introduction. The Dedekind sum can be defined for two relatively prime positive integers a, b by

$$\mathfrak{s}(a,b) = \frac{1}{4b} \sum_{k=1}^{b-1} \cot \frac{\pi k a}{b} \cot \frac{\pi k}{b}.$$

These sums appear in various branches of mathematics: number theory, algebraic geometry, and topology; they have consequently been studied extensively in various contexts. These include the quadratic reciprocity law [**13**], random number generators [**12**], group actions on complex manifolds [**9**], and lattice point problems ([**14**] or [**5**]). Dedekind was the first to show the following reciprocity law [**3**]:

$$\mathfrak{s}(a,b) + \mathfrak{s}(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) \tag{1}$$

He was led naturally to this reciprocity law by considering the η-function $\eta(\tau) = e^{\pi i \tau/12} \prod_{m=1}^{\infty}(1 - e^{2\pi i m \tau})$ on the complex upper half plane and transforming it under the action of the modular group $SL_2(\mathbb{Z})$.

Gauß's law of quadratic reciprocity, for example, follows easily from (1); see [**13**] or [**16**]. We note that $\mathfrak{s}(a,b) = \mathfrak{s}(a \bmod b, b)$. Combining this with the reciprocity law (1), one obtains a polynomial-time algorithm for computing $\mathfrak{s}(a,b)$, similar in spirit to the Euclidean algorithm. From this point of view, it is not surprising (though not obvious) that $\mathfrak{s}(a,b)$ can be expressed efficiently in terms of the continued fraction expansion of a/b; see [**8**] or [**19**].

Rademacher was one of the pioneers in the use of Dedekind sums [**17**]; in fact, he found several proofs of (1) [**16**]. We present yet another proof, which establishes a simple connection with lattice point enumeration in polytopes. The reciprocity law (1) follows readily once the reader is familiar with the computation of the coefficients of the Ehrhart polynomial for a lattice polytope.

2. COUNTING LATTICE POINTS. Let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the n-dimensional integer lattice, and let $\mathscr{P}$ be an n-dimensional lattice polytope in $\mathbb{R}^n$, so $\mathscr{P}$ is a compact simplicial complex of pure dimension n whose vertices lie on the lattice. For $t \in \mathbb{N}$, denote by $L(\mathscr{P}, t)$ the number of lattice points in the closure of the dilated polytope $t\mathscr{P} := \{tx : x \in \mathscr{P}\}$. Ehrhart proved that $L(\mathscr{P}, t)$ is a polynomial in t of degree n [**6**]. Moreover,

$$L(\mathscr{P}, t) = \mathrm{Vol}(\mathscr{P})t^n + \tfrac{1}{2}\mathrm{Vol}(\partial\mathscr{P})t^{n-1} + \cdots + \chi(\mathscr{P}).$$

Here, $\mathrm{Vol}(\partial\mathscr{P})$ denotes the surface area of $\mathscr{P}$ normalized with respect to the sublattice on each face of $\mathscr{P}$, and $\chi(\mathscr{P})$ is the Euler characteristic of $\mathscr{P}$. We note that, for convex polytopes $\mathscr{P}$, $\chi(\mathscr{P}) = 1$ [**6**].

In this paper, we focus on the case $\mathbb{R}^2$, where Ehrhart's result is known as Pick's Theorem; see [7] or [4]: For a convex lattice polytope $\mathscr{P} \in \mathbb{R}^2$,

$$L(\mathscr{P}, t) = At^2 + \tfrac{1}{2}Bt + 1,$$

where A is the area and B is the number of boundary lattice points of $\mathscr{P}$.

In the general case, the other coefficients of $L(\mathscr{P}, t)$ are not as easily accessible. In fact, until quite recently a method of computing these coefficients was unknown. There has been recent progress in this direction ([**1**], [**2**], [**10**], and [**11**]); Diaz and Robins found a way of proving a cotangent representation for the generating function $\sum_{t=0}^{\infty} L(\mathscr{P}, t)e^{-2\pi st}$, thereby deriving a formula for the Ehrhart coefficients of $L(\mathscr{P}, t)$ [**5**]. For our purposes, the following result (a straightforward consequence of [**5**, Corollary 1]) is sufficient:

Theorem. *Let $\mathscr{P}$ denote the simplex in $\mathbb{R}^n$ with the vertices $(0, \dots, 0)$, $(a_1, 0, \dots, 0)$, $(0, a_2, 0, \dots, 0), \dots, (0, \dots, 0, a_n)$, where $a_1, \dots, a_n \in \mathbb{N}$ are pairwise coprime. Denote the corresponding Ehrhart polynomial by $L(\mathscr{P}, t) = \sum_{j=0}^{n} c_j t^j$. Then c_m is the coefficient of $s^{-(m+1)}$ in the Laurent expansion at $s = 0$ of*

$$\frac{\pi^{m+1}}{m!\,2^{n-m}p} \sum_{r=1}^{p} \left(1 + \coth\frac{\pi}{a_1}(s + ir)\right)\left(1 + \coth\frac{\pi}{a_2}(s + ir)\right)$$
$$\cdots \left(1 + \coth\frac{\pi}{a_n}(s + ir)\right)\left(1 + \coth\frac{\pi}{p}(s + ir)\right),$$

where $p = a_1 \cdots a_n$.

The appearance of cotangent products in this result leads us to expect Dedekind sums in some form within the coefficients of the Ehrhart polynomial, thus also within the formulas for the number of lattice points in simplices. In fact, the nontrivial cases of dimension three [**15**] and four [**18**] involve classical Dedekind sums. Both formulas can be obtained easily through the Theorem.

We use this result in an indirect way. Precisely, we compute c_0 according to the Theorem, and make use of the fact that $c_0 = \chi(\mathscr{P}) = 1$. Dedekind's reciprocity law (1) follows from this idea if we consider the case of dimension $n = 2$.

3. PROOF OF THE RECIPROCITY LAW. According to the Theorem, for coprime a and b we have to find the coefficient of s^{-1} of the Laurent series at $s = 0$ of

$$\frac{\pi}{4ab} \sum_{r=1}^{ab} \left(1 + \coth\frac{\pi}{a}(s + ir)\right)\left(1 + \coth\frac{\pi}{b}(s + ir)\right)\left(1 + \coth\frac{\pi}{ab}(s + ir)\right). \quad (2)$$

The Laurent expansion of each factor depends on r:

$$1 + \coth\frac{\pi}{c}(s + ir) = \begin{cases} S_c := \dfrac{c}{\pi}s^{-1} + 1 + \dfrac{\pi}{3c}s + O(s^3) & \text{if } c \mid r \\ R_c := 1 + \coth\dfrac{\pi ir}{c} + O(s) & \text{if } c \nmid r \end{cases}$$

To keep track of the various cases, we introduce the notation

$$\chi_c = \begin{cases} 1 \text{ if } c \mid r \\ 0 \text{ if } c \nmid r, \end{cases}$$

so that we can write $1 + \coth\pi(s + ir)/c = S_c \chi_c + R_c(1 - \chi_c)$, and (2) becomes

$$\sum_{r=1}^{ab} (S_a \chi_a + R_a(1 - \chi_a))(S_b \chi_b + R_b(1 - \chi_b))(S_{ab} \chi_{ab} + R_{ab}(1 - \chi_{ab})).$$

Now, expand this into all 8 terms, and consider each summand according to the number of S_c factors:

1. Terms with one S_c factor are

$$\begin{aligned} S_a \chi_a R_b(1 - \chi_b) R_{ab}(1 - \chi_{ab}) &= S_a R_b R_{ab} \chi_a(1 - \chi_b - \chi_{ab} + \chi_{ab}) \\ &= S_a R_b R_{ab}(\chi_a - \chi_{ab}) \qquad (3) \end{aligned}$$

and, similarly,

$$R_a(1 - \chi_a) S_b \chi_b R_{ab}(1 - \chi_{ab}) = R_a S_b R_{ab}(\chi_b - \chi_{ab}). \qquad (4)$$

The summand with S_{ab} is zero (note that $\chi_a \chi_{ab} = \chi_b \chi_{ab} = \chi_{ab}$, and $\chi_a \chi_b = \chi_{ab}$). To compute the contribution of (3), note that the support of $\chi_a - \chi_{ab}$ in $\{1, \dots, ab\}$ is $\{ka : 1 \le k \le b - 1\}$; thus its contribution to (2) is

$$\begin{aligned} &\frac{\pi}{4ab} \cdot \frac{a}{\pi} \sum_{k=1}^{b-1} \left(1 + \coth\frac{\pi ika}{b}\right)\left(1 + \coth\frac{\pi ika}{ab}\right) \\ &= \frac{1}{4b} \sum_{k=1}^{b-1} \left(1 - i\cot\frac{\pi ka}{b}\right)\left(1 - i\cot\frac{\pi k}{b}\right) \\ &= \frac{1}{4b} \sum_{k=1}^{b-1} 1 - \cot\frac{\pi ka}{b}\cot\frac{\pi k}{b} + i \cdots = \frac{1}{4} - \frac{1}{4b} - \mathfrak{s}(a, b). \end{aligned}$$

The imaginary part in the preceding sum has to be zero, because the original generating function is real. Similarly, (4) gives a contribution of $\frac{1}{4} - \frac{1}{4}a^{-1} - \mathfrak{s}(b, a)$.

2. There are no terms with two S_c factors, because

$$S_a \chi_a S_b \chi_b R_{ab}(1 - \chi_{ab}) = S_a S_b R_{ab} \chi_{ab}(1 - \chi_{ab}) = 0$$

and

$$S_a \chi_a R_b(1 - \chi_b) S_{ab} \chi_{ab} = S_a R_b S_{ab} \chi_{ab}(1 - \chi_b) = 0.$$

3. Finally, the term $S_a \chi_a S_b \chi_b S_{ab} \chi_{ab} = S_a S_b S_{ab} \chi_{ab}$ has support $\{ab\}$, and gives a contribution of

$$\begin{aligned} &\frac{\pi}{4ab}\left(\frac{a}{\pi}\frac{b}{\pi}\frac{\pi}{3ab} + \frac{a}{\pi}\frac{ab}{\pi}\frac{\pi}{3b} + \frac{b}{\pi}\frac{ab}{\pi}\frac{\pi}{3a} + \frac{a}{\pi} + \frac{b}{\pi} + \frac{ab}{\pi}\right) \\ &= \frac{1}{12}\left(\frac{1}{ab} + \frac{a}{b} + \frac{b}{a}\right) + \frac{1}{4}\left(\frac{1}{b} + \frac{1}{a} + 1\right). \end{aligned}$$

Adding all contributions, we arrive at

$$1 = c_0 = \frac{3}{4} + \frac{1}{12}\left(\frac{1}{ab} + \frac{a}{b} + \frac{b}{a}\right) - \mathfrak{s}(a, b) - \mathfrak{s}(b, a),$$

the desired reciprocity law (1).

The same method applied to dimension $n = 3$ does not give any further results. However, for $n = 4$, higher dimensional Dedekind sums **[20]** appear within the computations, so that this case is likely to provide new results.

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REFERENCES

1. A. I. Barvinok, Computing the Ehrhart polynomial of a convex lattice polytype, *Discrete Comput. Geom.* 12 (1994) 35–48.
2. M. Brion and M. Vergne, Lattice points in simple polytypes, *J. Amer. Math. Soc.* 10 (1997) 371–392.
3. R. Dedekind, Erläuterungen zu den Fragmenten XXVIII, in *Collected Works of Bernhard Riemann*, Dover Publ,. New York, 1953, pp. 466–478.
4. R. Diaz and S. Robins, Pick's formula via the Weierstrass $\wp$-function, *Amer. Math. Monthly* 102 (1995) 431–437.
5. ______, and ______, The Erhart polynomial of a lattice polytype, *Annals of Math.* 145 (1997) 503–518.
6. E. Ehrhart, Sur un problème de géométrie diophantienne linéaire II, *J. reine angewandte Math.* 227 (1967) 25–49.
7. B. Grünbaum and G. C. Shephard, Pick's Theorem, *Amer. Math. Monthly* 100 (1993) 150–161.
8. D. Hickerson, Continued fractions and density results for Dedekind sums, *J. reine angewandte Math.* 290 (1977) 113–116.
9. F. Hirzebruch and D. Zagier, *The Atiya-Singer index theorem and elementary number theory*, Publish or Perish Press, Boston, 1974.
10. J. M. Kantor and A. G. Khovanskii, Une application du Théorème de Riemann-Roch combinatoire au polynôme d'Ehrhart des polytypes entier de $\mathbb{R}^n$, *C. R. Acad. Sci. Paris Series I* 317 (1993) 501–507.
11. A. G. Khovanskii and A. V. Pukhlikov, The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes, *St. Petersburg Math. J.* 4 (1993) 789–812.
12. D. Knuth, *The art of computer programming*, Vol. 2. Addison-Wesley, Reading, Mass., 1981.
13. C. Meyer, Über einge Anwendungen Dedekindscher Summen, *J. reine angewandte Math.* 198 (1957) 143–203.
14. J. Pommersheim, Toric varieties, lattice points, and Dedekind sums, *Math. Ann.* 295 (1993) 1–24.
15. H. Rademacher, On Dedekind sums and lattice points in a tetrahedron, *Collected papers of Hans Rademacher*, MIT Press, 1974, pp. 391–398.
16. ______, and E. Grosswald, *Dedekind sums*, Carus Mathematical Monographs, The Mathematical Association of America, 1972.
17. ______, and A. Whiteman, Theorems on Dedekind sums, *Amer. J. Math.* 63 (1941) 377–407.
18. K. Rosen, Lattice points in four-dimensional tetrahedra and a conjecture of Rademacher, *J. reine angewandte Math.* 307/308 (1979) 264–275.
19. I. Vardi, The distribution of Dedekind sums, preprint, 1992.
20. D. Zagier, Higher dimensional Dedekind sums, *Math. Ann.* 202 (1973) 149–172.

Author's comment: In the course of proofreading, it was discovered that Pommersheim made an observation in his paper **[14]** similar to our idea of equating the Euler characteristic with the given cotangent Laurent expansion. His approach used toric varieties but translates into an equivalent statement.

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