# PROBLEMS AND SOLUTIONS

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with the collaboration of Paul T. Bateman, Mario Benedicty, Paul Bracken, Duane M. Broline, Ezra A. Brown, Richard T. Bumby, Glenn G. Chappell, Randall Dougherty, Roger B. Eggleton, Ira M. Gessel, Bart Goddard, Jerrold R. Griggs, Douglas A. Hensley, Richard Holzsager, John R. Isbell, Robert Israel, Kiran S. Kedlaya, Murray S. Klamkin, Fred Kochman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Charles Vanden Eynden, and William E. Watkins.

*Proposed problems and solutions should be sent in duplicate to the* MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before October 31, 1999; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

## PROBLEMS

**10732.** *Proposed by M. N. Deshpande, Nagpur, India.* Let $n$ and $k$ be positive integers with $k < n$. Select a permutation $\pi$ of $n$ objects at random, and let the random variable $X_k$ denote the number of objects that lie in cycles of $\pi$ of length less than or equal to $k$. Find the expected value and the variance of $X_k$.

**10733.** *Proposed by Sung Soo Kim, Hanyang University, Ansan, Korea.* Let $\{E_\alpha\}_{\alpha\in\Omega}$ be a partition of the unit interval $I = [0, 1]$ into nonempty sets that are closed in the usual topology. Is it possible that

**(a)** $\Omega$ is uncountable and $E_\alpha$ is uncountable for each $\alpha \in \Omega$?
**(b)** $\Omega$ is uncountable but $E_\alpha$ is countably infinite for each $\alpha \in \Omega$?
**(c)** $\Omega$ is countably infinite?

**10734.** *Proposed by Floor van Lamoen, Goes, The Netherlands.* Let $ABC$ be a triangle with orthocenter $H$, incenter $I$, and circumcenter $O$. Let $[P, r]$ denote the circle with center $P$ and radius $r$. Show that the radical center of $[A, CA + AB]$, $[B, AB + BC]$, and $[C, BC + CA]$ is the point obtained by reflecting $H$ through $O$ and then reflecting the result through $I$.

**10735.** *Proposed by Gustavus J. Simmons, Sandia Park, NM.* If $L_n$ is the $n$-by-$n$ matrix with $i, j$-entry equal to $\binom{i-1}{j-1}$, then $L_n^2 \equiv I_n \bmod 2$, where $I_n$ is the $n$-by-$n$ identity matrix. Show that if $R_n$ is the $n$-by-$n$ matrix with $i, j$-entry equal to $\binom{i-1}{n-j}$, then $R_n^3 \equiv I_n \bmod 2$.

**10736.** *Proposed by Mizan R. Khan, Eastern Connecticut State University, Willimantic, CT.* For a given $n \geq 2$, let $M(n) = \max\{\,|a - b| : a, b \in \{1, 2, \ldots, n\} \text{ and } ab \equiv 1 \bmod n\,\}$.

**(a)** Find a closed-form expression $U(n)$ such that $M(n) \leq U(n)$ for all $n$, with equality in infinitely many cases.
**(b)** Show that $\lim_{n\to\infty} M(n)/n = 1$.
**(c)*** Prove or disprove that $\lim_{n\to\infty} \log(n - M(n))/\log n = 1/2$.