

10738



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10737. Proposed by Hassan Ali Shah Ali, Tehran, Iran. Let m and n be positive integers with $n \geq 2m$, and let $a_1 \leq a_2 \leq \dots \leq a_n$ be positive integers such that

$$a_n < m + \frac{1}{2m} \left(\sum_{i=1}^m \binom{n}{2i} \binom{2i}{i} \right).$$

Show that there exist two different n -tuples $(\epsilon_1, \dots, \epsilon_n)$ and $(\delta_1, \dots, \delta_n)$, with entries 0, 1, and 2, such that $\sum_{j=1}^n \epsilon_j = \sum_{j=1}^n \delta_j \leq 2m$ and $\sum_{j=1}^n \epsilon_j a_j = \sum_{j=1}^n \delta_j a_j$.

10738. Proposed by Radu Theodorescu, Université Laval, Sainte-Foy, PQ, Canada. For $t > 0$, let $m_n(t) = \sum_{k=0}^{\infty} k^n e^{-t} t^k / k!$ be the n th moment of a Poisson distribution with parameter t . Let $c_n(t) = m_n(t) / n!$. A sequence a_0, a_1, \dots is *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n > 0$ and is *log-concave* if $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n > 0$.

(a) Show that $m_0(t), m_1(t), \dots$ is log-convex.

(b) Show that $c_0(t), c_1(t), \dots$ is not log-concave when $t < 1$.

(c) Show that $c_0(t), c_1(t), \dots$ is log-concave when t is sufficiently large.

(d)* Is $c_0(t), c_1(t), \dots$ log-concave when $t \geq 1$?

SOLUTIONS

Moments of Roots of Chebyshev Polynomials

10448 [1995, 360]. Proposed by Fu-Chuen Chang, National Sun Yat-sen University, Kaohsiung, Taiwan. Fix a positive integer n . Let $x_i = \cos((2i-1)\pi/(2n))$ for $1 \leq i \leq n$, and let $c_k = \frac{1}{n} \sum_{i=1}^n x_i^k$ for $k \in \mathbb{N}$. Show that

$$c_k = \begin{cases} 0 & \text{if } k = 1, 3, \dots, 2n-1; \\ \binom{k}{k/2} 2^{-k} & \text{if } k = 0, 2, \dots, 2n-2. \end{cases}$$

Solution 1 by Paul Deiermann, Louisiana State University, Shreveport, LA. When $k = 0$ and n is odd, the term for $j = (n+1)/2$ appears as 0^0 , which must be taken to be 1 to arrive at the stated formula and our generalization. We show, for arbitrary integers $k \geq 0$, that

$$c_k = \begin{cases} 0 & \text{for } k \text{ odd,} \\ 2^{-k} \sum_{p=-m}^m (-1)^p \binom{k}{pn + \frac{k}{2}} & \text{for } k \text{ even,} \end{cases}$$

where $m = \lfloor k/(2n) \rfloor$. The stated problem covers those k for which $m = 0$.

First note that $x_{n+1-j} = -x_j$, so the terms of the sum cancel in pairs when k is odd. We may thus restrict to the case of k even. Since $x_j = (e^{i\pi(2j-1)/(2n)} + e^{-i\pi(2j-1)/(2n)})/2$, the binomial theorem and a summation of a finite geometric progression imply

$$\begin{aligned} \sum_{j=1}^n x_j^k &= \sum_{j=1}^n 2^{-k} \left(e^{i\pi \frac{2j-1}{2n}} + e^{-i\pi \frac{2j-1}{2n}} \right)^k = 2^{-k} \sum_{j=1}^n \sum_{q=0}^k \binom{k}{q} e^{i \frac{\pi}{2n} (k-2q)} e^{i \frac{2\pi}{n} (q-k/2)j} \\ &= 2^{-k} \sum_{q=0}^k \binom{k}{q} e^{i \frac{\pi}{2n} (k-2q)} \sum_{j=1}^n e^{i \frac{2\pi}{n} (q-k/2)j} = 2^{-k} \sum_{q=0}^k \binom{k}{q} e^{i \frac{\pi}{2n} (2q-k)} \sum_{u=0}^{n-1} e^{i \frac{2\pi}{n} (q-k/2)u} \\ &= 2^{-k} \sum_{q=0}^k \binom{k}{q} e^{i \frac{\pi}{2n} (2q-k)} \begin{cases} n & \text{if } q - k/2 = pn, p \in \mathbb{Z}, \\ \frac{1 - e^{i\pi(2q-k)}}{1 - e^{i \frac{2\pi}{n} (q-k/2)}} = 0 & \text{if } n \nmid q - k/2. \end{cases} \end{aligned}$$

Since k is even, $q - k/2 = pn$ implies $q = pn + k/2$. Then, $0 \leq q \leq k$ gives $-m \leq p \leq m$. Also, in this case, $e^{i \frac{\pi}{2n} (2q-k)} = e^{i\pi p} = (-1)^p$. Thus, we get

$$\sum_{j=1}^n x_j^k = 2^{-k} n \sum_{p=-m}^m (-1)^p \binom{k}{pn + \frac{k}{2}}.$$