10737. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.* Let m and n be positive integers with $n \geq 2m$, and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers such that

$$a_n < m + \frac{1}{2m}\left(\sum_{i=1}^{m}\binom{n}{2i}\binom{2i}{i}\right).$$

Show that there exist two different n-tuples $(\epsilon_1, \ldots, \epsilon_n)$ and $(\delta_1, \ldots, \delta_n)$, with entries 0, 1, and 2, such that $\sum_{j=1}^{n} \epsilon_j = \sum_{j=1}^{n} \delta_j \leq 2m$ and $\sum_{j=1}^{n} \epsilon_j a_j = \sum_{j=1}^{n} \delta_j a_j$.

10738. *Proposed by Radu Theodorescu, Université Laval, Sainte-Foy, PQ, Canada.* For $t > 0$, let $m_n(t) = \sum_{k=0}^{\infty} k^n e^{-t} t^k / k!$ be the nth moment of a Poisson distribution with parameter t. Let $c_n(t) = m_n(t)/n!$. A sequence $a_0, a_1, \ldots$ is *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n > 0$ and is *log-concave* if $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n > 0$.
(**a**) Show that $m_0(t), m_1(t), \ldots$ is log-convex.
(**b**) Show that $c_0(t), c_1(t), \ldots$ is not log-concave when $t < 1$.
(**c**) Show that $c_0(t), c_1(t), \ldots$ is log-concave when t is sufficiently large.
(**d**)* Is $c_0(t), c_1(t), \ldots$ log-concave when $t \geq 1$?

SOLUTIONS

Moments of Roots of Chebyshev Polynomials

10448 [1995, 360]. *Proposed by Fu-Chuen Chang, National Sun Yat-sen University, Kaohsiung, Taiwan.* Fix a positive integer n. Let $x_i = \cos\big((2i-1)\pi/(2n)\big)$ for $1 \leq i \leq n$, and let $c_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$ for $k \in \mathbb{N}$. Show that

$$c_k = \begin{cases} 0 & \text{if } k = 1, 3, \ldots, 2n-1; \\ \binom{k}{k/2} 2^{-k} & \text{if } k = 0, 2, \ldots, 2n-2. \end{cases}$$

Solution I by Paul Deiermann, Louisiana State University, Shreveport, LA. When $k = 0$ and n is odd, the term for $j = (n+1)/2$ appears as 0^0, which must be taken to be 1 to arrive at the stated formula and our generalization. We show, for arbitrary integers $k \geq 0$, that

$$c_k = \begin{cases} 0 & \text{for } k \text{ odd}, \\ 2^{-k}\sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}} & \text{for } k \text{ even}, \end{cases}$$

where $m = \lfloor k/(2n) \rfloor$. The stated problem covers those k for which $m = 0$.

First note that $x_{n+1-j} = -x_j$, so the terms of the sum cancel in pairs when k is odd. We may thus restrict to the case of k even. Since $x_j = \left(e^{i\pi(2j-1)/(2n)} + e^{-i\pi(2j-1)/(2n)}\right)/2$, the binomial theorem and a summation of a finite geometric progression imply

$$\sum_{j=1}^{n} x_j^k = \sum_{j=1}^{n} 2^{-k}\left(e^{i\pi\frac{2j-1}{2n}} + e^{-i\pi\frac{2j-1}{2n}}\right)^k = 2^{-k}\sum_{j=1}^{n}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(k-2q)} e^{i\frac{2\pi}{n}(q-k/2)j}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(k-2q)}\sum_{j=1}^{n} e^{i\frac{2\pi}{n}(q-k/2)j} = 2^{-k}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(2q-k)}\sum_{u=0}^{n-1} e^{i\frac{2\pi}{n}(q-k/2)u}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(2q-k)} \begin{cases} n & \text{if } q - k/2 = pn,\ p \in \mathbb{Z}, \\ \frac{1-e^{i\pi(2q-k)}}{1-e^{i\frac{2\pi}{n}(q-k/2)}} = 0 & \text{if } n \nmid q - k/2. \end{cases}$$

Since k is even, $q - k/2 = pn$ implies $q = pn + k/2$. Then, $0 \leq q \leq k$ gives $-m \leq p \leq m$. Also, in this case, $e^{i\frac{\pi}{2n}(2q-k)} = e^{i\pi p} = (-1)^p$. Thus, we get

$$\sum_{j=1}^{n} x_j^k = 2^{-k} n \sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}}.$$

Solution II by Walter Van Assche, Katholieke Universiteit Leuven, Heverlee, Belgium. The x_i are the zeros of the Chebyshev polynomial of the first kind T_n of degree n. The Gauss-Chebyshev quadrature formula has the property that the quadrature weights are constant; thus Gaussian quadrature gives

$$\frac{1}{n}\sum_{k=1}^{n} f(x_i) = \frac{1}{\pi}\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}}$$

for every polynomial f of degree at most $2n-1$ (T. J. Rivlin, *Chebyshev Polynomials*, Wiley, 1990, pp. 43–46). Taking $f(x) = x^k$ for $0 \le k \le 2n-1$ then gives

$$c_k = \frac{1}{\pi}\int_{-1}^{1} x^k \frac{dx}{\sqrt{1-x^2}}.$$

By symmetry this integral vanishes when k is odd. When k is even, the symmetry and the substitution $x^2 = t$ gives

$$\int_{-1}^{1} x^k \frac{dx}{\sqrt{1-x^2}} = \int_{0}^{1} t^{\frac{k-1}{2}} \frac{dt}{\sqrt{1-t}}.$$

The latter is Euler's Beta function $B((k+1)/2, 1/2) = \Gamma((k+1)/2)\Gamma(1/2)/\Gamma(k/2+1)$. Now use Legendre's duplication formula $\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2)$ with $2z = k+1$ and $\Gamma(1/2) = \sqrt{\pi}$ to find the desired results.

Solution III by Franz Peherstorfer, Johannes Kepler Universität, Linz, Austria. For $x \in [-1,1]$, let $T_n(x) = \cos(n \arccos x)$ and $U_n(x) = \sin((n+1)\arccos x)/\sin(\arccos x)$ denote the degree n Chebyshev polynomials of the first and second kind, respectively. Since $T_n(x) = 2^{n-1}\prod_{i=1}^{n}(x-x_i)$ and $T_n'(x) = nU_{n-1}(x)$, we have

$$\frac{U_{n-1}(x)}{T_n(x)} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x-x_i} = \sum_{k=0}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^n\right)\frac{1}{x^{k+1}} \tag{1}$$

for $|x| > 1$, where the second equality follows from a series expansion of $(1-x_i/x)^{-1}$.

On the other hand, we have $T_n^2(x) - (x^2-1)U_{n-1}^2(x) = 1$ for all $x \in \mathbb{R}$. Dividing both sides of this equation by $(x^2-1)T_n^2(x)$ gives

$$\left(\frac{1}{\sqrt{x^2-1}} - \frac{U_{n-1}(x)}{T_n(x)}\right)\left(\frac{1}{\sqrt{x^2-1}} + \frac{U_{n-1}(x)}{T_n(x)}\right) = \frac{1}{(x^2-1)T_n^2(x)} = O\left(\frac{1}{x^{2n+2}}\right)$$

as $x \to \infty$. Since $\lim_{x\to\infty} xU_{n-1}(x)/T_n(x) = 1$, this implies

$$\frac{U_{n-1}(x)}{T_n(x)} = \frac{1}{\sqrt{x^2-1}} + O\left(\frac{1}{x^{2n+1}}\right) \tag{2}$$

as $x \to \infty$. Taking the series expansion of $\sqrt{1-x^{-2}}$ in (2) and comparing to the series in (1) gives the desired result.

Editorial comment. Wolfdieter Lang noted that the generating function $\sum_{k\ge 0} c_k z^k$ has been computed explicitly as an elementary function. See W. Lang, On sums of powers of zeros of polynomials, *J. Comp. Appl. Math.* 88 (1998) 237–256 for details and further references.

Solved also by U. Abel (Germany), J. Anglesio (France), G. Bach (Germany), K. L. Bernstein, N. Bhatnagar, J. C. Binz (Switzerland), P. Bracken & S. Dorf (Canada), R. J. Chapman (U. K.), H. Chen, E. Cohen (France), D. A. Darling, K. Diethelm (Germany), C. J. Efthimiou, R. Ehrenborg (Canada), S. M. Gagola Jr., M. E. H. Ismail, N. Komanda, R. L. Lamphere, W. Lang (Germany), J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Pedersen (Denmark), N. Rosenberg, K. Foltz, H.-J. Seiffert (Germany), S. J. Smith (Australia), A. Stenger, R. Stong, M. Vowe (Switzerland), H. Widmer (Switzerland), Anchorage Math Solutions Group, NSA Problems Group, and the proposer.