n-Tuples Whose Elements Divide Their Sum: 10597

David Cox; GCHQ Problem Solving Group

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Since $r_i = (1+k+k^2+b_i)/2$ and r_i is an integer, b_i must be odd, and so $1-b_i^2 \le 0$ for all i. Therefore $\sum_{i=1}^m b_i \le 0$. The total number of positive entries in the matrix thus satisfies

$$\sum_{i=1}^m r_i = \sum_{i=1}^m \frac{1}{2}(n+k+b_i) = \frac{m}{2}(n+k) + \frac{1}{2}\sum_{i=1}^m b_i \le \frac{m}{2}(n+k).$$

Achieving the bound requires $\sum_{i=1}^m b_i = 0$, which occurs only when half the rows have $b_i = +1$ and the other half have $b_i = -1$. Thus it is necessary that m be even. One matrix that achieves the bound when $m = 2(n!)$ is formed by taking all $n!$ permutations of a row with $\frac{1}{2}(n+k+1)$ positive entries and all $n!$ permutations of a row with $\frac{1}{2}(n+k-1)$ positive entries. By symmetry, all of the dot products are equal, and their sum is zero; hence each dot product must be zero.

Editorial comment. John H. Lindsey observed that equality in the bound requires m to be divisible by 4. The proposer asked for the smallest number of rows allowing equality to be achieved for a given n. He and Richard Stong independently provided a construction with $m = 2\binom{n-1}{(k+1)k/2}$.

Solved also by R. J. Chapman (U. K.), J. H. Lindsey II, K. McInturff, R. Stong, and the proposer.

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10597 [1997, 457]. *Proposed by David Cox, Amherst College, Amherst, MA.* Fix an integer $n \ge 2$, and let $d_1, d_2, \dots, d_n$ be positive integers with no common divisor greater than 1. Suppose that d_i divides $d_1 + \cdots + d_n$ for $1 \le i \le n$.
(a) Prove that $d_1 d_2 \cdots d_n$ divides $(d_1 + \cdots + d_n)^{n-2}$.
(b) For each $n \ge 3$, give an example to show that the exponent in part **(a)** cannot be made smaller.

Solution by GCHQ Problems Group, Cheltenham, U. K.
(a) Let p be a prime factor of the product $d_1 d_2 \cdots d_n$, and let p^k be the highest power of p dividing any one of the d_i. We have $p^k \mid \sum d_i$, and thus $p^{k(n-2)} \mid \left(\sum d_i\right)^{n-2}$. Since $d_1, \dots, d_n$ have no common factor greater than 1, some element d_j is not divisible by p. Furthermore, since $p \mid \sum d_i$, at least two summands are not divisible by p. Hence the highest power of p dividing $\prod d_i$ does not exceed $p^{k(n-2)}$. Repeating this for each prime factor shows that $\prod d_i$ divides $(\sum d_i)^{n-2}$.
(b) Let $d_1 = 1$, $d_2 = n-1$, and $d_i = n$ for $3 \le i \le n$. Here $\sum d_i = n(n-1)$, which is divisible by each d_i. Since $d_1 = 1$, the greatest common divisor is 1. We have $\prod d_i = n^{n-2}(n-1)$. Since n and $n-1$ are coprime, the smallest power of $n(n-1)$ divisible by $n^{n-2}(n-1)$ is $(n(n-1))^{n-2}$, and thus the exponent cannot be reduced.

Editorial comment. Other examples submitted for part **(b)** by various solvers included

$$d_1 = 1, d_2 = 2, \text{ and } d_i = 3 \cdot 2^{i-3} \text{ for } 3 \le i \le n$$

and

$$d_1 = 1, d_i = 2 \text{ for } 2 \le i \le n-1, \text{ and } d_n = 2n-3.$$

Using Euclid's sequence 2, 3, 7, 43, 1807, ..., the San Jose State Problem Solving Ring gave an example in which $d_1 d_2 \cdots d_n = (d_1 + \cdots + d_n)^{n-2}$. Another use of Euclid's sequence appears in this MONTHLY in the solution of Problem 10532 [1996, 510; 1998, 775], where references are given.

M. J. Knight and the San Jose State Problem Solving Ring each showed that for given n the set D_n of n-tuples $(d_1, d_2, \dots, d_n)$ satisfying the conditions of the problem is finite. For example, D_2 contains only the pair (1, 1), and D_3 contains only the triples (1, 1, 1), (1, 1, 2), (1, 2, 3), and their permutations. The finiteness of D_n is equivalent to the finiteness of the set X_n of solutions of $1/x_1 + 1/x_2 + \cdots + 1/x_n = 1$ in positive integers, which was apparently first established by D. R. Curtiss, this MONTHLY 29 (1922) 380–387. A direct bijection between D_n and X_n is obtained by setting $x_j = \left(\sum d_i\right)/d_j$.

Solved also by R. Barbara (Lebanon), D. Beckwith, M. Boase (U.K.), J. Brawner, D. Callan, R. J. Chapman (U. K.), T. Hermann, R. Holzsager, T. Jager, S. A. Jassim (U. K.), M. J. Knight, C. Lanski, J. H. Lindsey II, D. Lorenzini, K. McInturff, R. Padma (India), K. Schilling, R. Stong, A. Tissier (France), SJSU Problem Solving Ring, and the proposer.

Binomial Ratios

10625 [1997, 871]. *Proposed by Olaf Krafft and Martin Schaefer, Technical University Aachen, Aachen, Germany.* For $x > 0$ and $n \in \mathbb{N}$, define

$$a_n = \sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} x^i \Bigg/ \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} x^i.$$

Evaluate $\lim_{n\to\infty} a_n$.

Solution I by Nora Thornber, Raritan Valley Community College, Somerville, NJ. Applying the binomial theorem four times, we have

$$a_n = \sqrt{x} \cdot \frac{\left(1+\sqrt{x}\right)^{2^n} + \left(1-\sqrt{x}\right)^{2^n}}{\left(1+\sqrt{x}\right)^{2^n} - \left(1-\sqrt{x}\right)^{2^n}} = \sqrt{x} \cdot \frac{1 + \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{2^n}}{1 - \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{2^n}}.$$

But $\left|(1-\sqrt{x})/(1+\sqrt{x})\right| < 1$, so we conclude that $\lim_{n\to\infty} a_n = \sqrt{x}$.

Solution II by The National Security Agency Problems Group, Fort Meade, MD. Let $p = \sqrt{x}/(\sqrt{x}+1)$ and $q = 1/(\sqrt{x}+1)$, so that $0 < p, q < 1$, $p + q = 1$, and $\sqrt{x} = p/q$. Now consider an experiment consisting of 2^n independent tosses of a coin that is biased to come up heads with probability p. Let E_n (respectively, O_n) be the probability that an even (respectively, odd) number of heads comes up. Set $u_n = u_n(p) = E_n/O_n$. Then

$$\begin{aligned} u_n &= \frac{\sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} p^{2i} q^{2^n-2i}}{\sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} p^{2i+1} q^{2^n-(2i+1)}} \\ &= \frac{q^{2^n} \sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} (p/q)^{2i}}{q^{2^n} \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} (p/q)^{2i+1}} = \frac{\sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} x^i}{\sqrt{x} \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} x^{2i+1}}. \end{aligned}$$

Hence $a_n = \sqrt{x} u_n$.

The independence of the various tosses implies $E_{n+1} = E_n E_n + O_n O_n$ and $O_{n+1} = 2E_n O_n$. Therefore

$$u_{n+1} = \frac{E_n^2 + O_n^2}{2E_n O_n} = \frac{1}{2}\left(u_n + \frac{1}{u_n}\right).$$

By the arithmetic-geometric mean inequality, $u_n \geq 1$; hence $u_n \geq (1/2)(u_n + 1/u_n) = u_{n+1}$. Therefore the sequence u_n is decreasing and bounded below; it follows that $L = \lim_{n\to\infty} u_n$ exists, and satisfies $L = (1/2)(L + 1/L)$. Therefore $L = 1$, so we conclude that $\lim_{n\to\infty} a_n = \sqrt{x}$.

Solution III by Ulrich Abel, Fachhochschule Giessen-Friedberg, Friedberg, Germany. We prove the following generalization: For integers $k \geq 1$, $r, s \geq 0$, and real $x > 0$, we have

$$b_n = \sum_{i\geq 0} \binom{kn}{ki+r} x^i \Bigg/ \sum_{i\geq 0} \binom{kn}{ki+s} x^i \longrightarrow x^{(s-r)/k}.$$

In the special case $k = 2$, $r = 0$, $s = 1$, we have $b_{2^{n-1}} = a_n$, and conclude that $a_n \to \sqrt{x}$.