of the set X_n of solutions of $1/x_1 + 1/x_2 + \cdots + 1/x_n = 1$ in positive integers, which was apparently first established by D. R. Curtiss, this MONTHLY 29 (1922) 380–387. A direct bijection between D_n and X_n is obtained by setting $x_j = \left(\sum d_i\right)/d_j$.

Solved also by R. Barbara (Lebanon), D. Beckwith, M. Boase (U.K.), J. Brawner, D. Callan, R. J. Chapman (U. K.), T. Hermann, R. Holzsager, T. Jager, S. A. Jassim (U. K.), M. J. Knight, C. Lanski, J. H. Lindsey II, D. Lorenzini, K. McInturff, R. Padma (India), K. Schilling, R. Stong, A. Tissier (France), SJSU Problem Solving Ring, and the proposer.

Binomial Ratios

10625 [1997, 871]. *Proposed by Olaf Krafft and Martin Schaefer, Technical University Aachen, Aachen, Germany.* For $x > 0$ and $n \in \mathbb{N}$, define

$$a_n = \sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} x^i \Big/ \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} x^i.$$

Evaluate $\lim_{n\to\infty} a_n$.

Solution I by Nora Thornber, Raritan Valley Community College, Somerville, NJ. Applying the binomial theorem four times, we have

$$a_n = \sqrt{x} \cdot \frac{\left(1+\sqrt{x}\right)^{2^n} + \left(1-\sqrt{x}\right)^{2^n}}{\left(1+\sqrt{x}\right)^{2^n} - \left(1-\sqrt{x}\right)^{2^n}} = \sqrt{x} \cdot \frac{1 + \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{2^n}}{1 - \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{2^n}}.$$

But $\left|(1-\sqrt{x})/(1+\sqrt{x})\right| < 1$, so we conclude that $\lim_{n\to\infty} a_n = \sqrt{x}$.

Solution II by The National Security Agency Problems Group, Fort Meade, MD. Let $p = \sqrt{x}/(\sqrt{x}+1)$ and $q = 1/(\sqrt{x}+1)$, so that $0 < p, q < 1$, $p + q = 1$, and $\sqrt{x} = p/q$. Now consider an experiment consisting of 2^n independent tosses of a coin that is biased to come up heads with probability p. Let E_n (respectively, O_n) be the probability that an even (respectively, odd) number of heads comes up. Set $u_n = u_n(p) = E_n/O_n$. Then

$$\begin{aligned} u_n &= \frac{\sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} p^{2i} q^{2^n-2i}}{\sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} p^{2i+1} q^{2^n-(2i+1)}} \\ &= \frac{q^{2^n} \sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} (p/q)^{2i}}{q^{2^n} \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} (p/q)^{2i+1}} = \frac{\sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} x^i}{\sqrt{x} \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} x^{2i+1}}. \end{aligned}$$

Hence $a_n = \sqrt{x} u_n$.

The independence of the various tosses implies $E_{n+1} = E_n E_n + O_n O_n$ and $O_{n+1} = 2E_n O_n$. Therefore

$$u_{n+1} = \frac{E_n^2 + O_n^2}{2E_n O_n} = \frac{1}{2}\left(u_n + \frac{1}{u_n}\right).$$

By the arithmetic-geometric mean inequality, $u_n \geq 1$; hence $u_n \geq (1/2)(u_n + 1/u_n) = u_{n+1}$. Therefore the sequence u_n is decreasing and bounded below; it follows that $L = \lim_{n\to\infty} u_n$ exists, and satisfies $L = (1/2)(L + 1/L)$. Therefore $L = 1$, so we conclude that $\lim_{n\to\infty} a_n = \sqrt{x}$.

Solution III by Ulrich Abel, Fachhochschule Giessen-Friedberg, Friedberg, Germany. We prove the following generalization: For integers $k \geq 1$, $r, s \geq 0$, and real $x > 0$, we have

$$b_n = \sum_{i\geq 0} \binom{kn}{ki+r} x^i \Big/ \sum_{i\geq 0} \binom{kn}{ki+s} x^i \longrightarrow x^{(s-r)/k}.$$

In the special case $k = 2$, $r = 0$, $s = 1$, we have $b_{2^{n-1}} = a_n$, and conclude that $a_n \to \sqrt{x}$.

Let z be a primitive kth root of unity. Then the finite geometric sum $\sum_{j=0}^{k-1} z^{ij}$ is k if i is a multiple of k and 0 otherwise. Choose $y > 0$ with $y^k = x$. We obtain

$$\sum_{i\geq 0}\binom{kn}{ki+r}x^i = \frac{1}{k}\sum_{i\geq 0}\binom{kn}{i+r}y^i\sum_{j=0}^{k-1}z^{ij} = \frac{1}{ky^r}\sum_{j=0}^{k-1}z^{-rj}\sum_{i\geq r}\binom{kn}{i}y^i z^{ij}$$
$$= \frac{1}{ky^r}\sum_{j=0}^{k-1}z^{-rj}(1+yz^j)^{kn} + O(n^{r-1}) = \frac{(1+y)^{kn}}{ky^r}\big(1+o(1)\big)$$

as $n \to \infty$, and this identity also holds with s in place of r. Therefore $b_n \to y^{s-r} = x^{(s-r)/k}$ as $n \to \infty$.

Editorial comment. Jean Anglesio noted that when x is a complex number (but not a negative real) the limit is the principal value of the square root of x. When $x < 0$ the limit does not exist.

Solved also by S. A. Ali, K. F. Andersen (Canada), J. Anglesio (France), D. Beckwith, C. Berg (Sweden), J. C. Binz (Switzerland), P. Bracken (Canada), D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), M. N. Deshpande (India), Z. Franco, C. Georghiou (Greece), T. Hermann, V. Hernandez (Spain), J.-H. Kim, R. A. Kopas, O. Kuba (Syria), N. F. Lindquist, J. H. Lindsey II, N. Lord (U. K.), S. Mahajan, D. A. Morales (Venezuela), M. Omarjee (France), M. M. Patnaik, G. Peng, H. Qin, H. Salle (The Netherlands), V. Schindler (Germany), R. Shahidi (Canada), N. C. Singer, A. Sofo (Australia), A. Stenger, D. B. Tyler, M. Vowe (Switzerland), M. Woltermann, Anchorage Math Solutions Group, GCHQ Problems Group, WMC Problems Group, and the proposer.

A Triangle Inequality

10644 [1998, 175]. *Proposed by Mihàly Bencze, Brazov, Romania.* Given an acute triangle with sides of length a, b, and c, inradius r, and circumradius R, prove that

$$\frac{r}{2R} \leq \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}}$$

Solution by the GCHQ Problems Group, Cheltenham, England. We have

$$a^2-(b^2+c^2)(1-\cos A) = b^2+c^2-2bc\cos A-(b^2+c^2)+(b^2+c^2)\cos A$$
$$= (b-c)^2\cos A \geq 0,$$

since A is acute. Hence $a^2 \geq (b^2+c^2)(1-\cos A) = 2(b^2+c^2)\sin^2(A/2)$. It follows that $a^2b^2c^2 \geq 8(a^2+b^2)(b^2+c^2)(c^2+a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)$, and so

$$\frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \geq 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

The standard fact $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$ now yields the required result.

Editorial comment. Several solvers noted that equality holds when the triangle is equilateral and that the result is valid also when the triangle is not acute.

Solved also by J. Anglesio (France), E. Braune (Austria), Z. Čerin (Croatia), J. Melville (Scotland), C. A. Minh, P. E. Nüesch (Switzerland), G. Peng, C. Popescu (Belgium), C. R. Pranesachar (India), S. M. Soltuz (Romania), M. Vowe (Switzerland), R. L. Young, SAS Maths Club (India), and the proposer.

Limit of a Recurrence

10648 [1998, 271]. *Proposed by N. P. Bhatia, University of Maryland, Baltimore County, MD, and W. O. Egerland, Bel Air, MD.* Let $z_1, z_2, \ldots, z_m$ be $m \geq 2$ points in the complex plane, and let $p_1, p_2, \ldots, p_m$ be positive real numbers such that $p_1 + p_2 + \cdots + p_m = 1$. For ω real and $n > m$, let $z_n = (p_1 z_{n-1} + p_2 z_{n-2} + \cdots + p_m z_{n-m})e^{i\omega}$. Show that the sequence $z_1, z_2, \ldots$ converges, and determine its limit.