Let z be a primitive kth root of unity. Then the finite geometric sum $\sum_{j=0}^{k-1} z^{ij}$ is k if i is a multiple of k and 0 otherwise. Choose $y > 0$ with $y^k = x$. We obtain

$$\sum_{i\geq 0}\binom{kn}{ki+r}x^i = \frac{1}{k}\sum_{i\geq 0}\binom{kn}{i+r}y^i\sum_{j=0}^{k-1}z^{ij} = \frac{1}{ky^r}\sum_{j=0}^{k-1}z^{-rj}\sum_{i\geq r}\binom{kn}{i}y^i z^{ij}$$
$$= \frac{1}{ky^r}\sum_{j=0}^{k-1}z^{-rj}(1+yz^j)^{kn} + O(n^{r-1}) = \frac{(1+y)^{kn}}{ky^r}\big(1+o(1)\big)$$

as $n \to \infty$, and this identity also holds with s in place of r. Therefore $b_n \to y^{s-r} = x^{(s-r)/k}$ as $n \to \infty$.

Editorial comment. Jean Anglesio noted that when x is a complex number (but not a negative real) the limit is the principal value of the square root of x. When $x < 0$ the limit does not exist.

Solved also by S. A. Ali, K. F. Andersen (Canada), J. Anglesio (France), D. Beckwith, C. Berg (Sweden), J. C. Binz (Switzerland), P. Bracken (Canada), D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), M. N. Deshpande (India), Z. Franco, C. Georghiou (Greece), T. Hermann, V. Hernandez (Spain), J.-H. Kim, R. A. Kopas, O. Kuba (Syria), N. F. Lindquist, J. H. Lindsey II, N. Lord (U. K.), S. Mahajan, D. A. Morales (Venezuela), M. Omarjee (France), M. M. Patnaik, G. Peng, H. Qin, H. Salle (The Netherlands), V. Schindler (Germany), R. Shahidi (Canada), N. C. Singer, A. Sofo (Australia), A. Stenger, D. B. Tyler, M. Vowe (Switzerland), M. Woltermann, Anchorage Math Solutions Group, GCHQ Problems Group, WMC Problems Group, and the proposer.

A Triangle Inequality

10644 [1998, 175]. *Proposed by Mihàly Bencze, Brazov, Romania.* Given an acute triangle with sides of length a, b, and c, inradius r, and circumradius R, prove that

$$\frac{r}{2R} \leq \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}}$$

Solution by the GCHQ Problems Group, Cheltenham, England. We have

$$a^2 - (b^2+c^2)(1-\cos A) = b^2 + c^2 - 2bc\cos A - (b^2+c^2) + (b^2+c^2)\cos A$$
$$= (b-c)^2\cos A \geq 0,$$

since A is acute. Hence $a^2 \geq (b^2+c^2)(1-\cos A) = 2(b^2+c^2)\sin^2(A/2)$. It follows that $a^2b^2c^2 \geq 8(a^2+b^2)(b^2+c^2)(c^2+a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)$, and so

$$\frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \geq 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

The standard fact $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$ now yields the required result.

Editorial comment. Several solvers noted that equality holds when the triangle is equilateral and that the result is valid also when the triangle is not acute.

Solved also by J. Anglesio (France), E. Braune (Austria), Z. Čerin (Croatia), J. Melville (Scotland), C. A. Minh, P. E. Nüesch (Switzerland), G. Peng, C. Popescu (Belgium), C. R. Pranesachar (India), S. M. Soltuz (Romania), M. Vowe (Switzerland), R. L. Young, SAS Maths Club (India), and the proposer.

Limit of a Recurrence

10648 [1998, 271]. *Proposed by N. P. Bhatia, University of Maryland, Baltimore County, MD, and W. O. Egerland, Bel Air, MD.* Let $z_1, z_2, \ldots, z_m$ be $m \geq 2$ points in the complex plane, and let $p_1, p_2, \ldots, p_m$ be positive real numbers such that $p_1 + p_2 + \cdots + p_m = 1$. For ω real and $n > m$, let $z_n = (p_1z_{n-1} + p_2z_{n-2} + \cdots + p_mz_{n-m})e^{i\omega}$. Show that the sequence $z_1, z_2, \ldots$ converges, and determine its limit.