A Matrix Representation for Euler's Constant, γ

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Euler's constant, $\gamma = 0.5772156649\ldots$ can be represented as the product of an infinite row vector, the inverse of a $\mathbb{Z}^+ \times \mathbb{Z}^+$ lower triangular matrix, and an infinite $\mathbb{Z}^+ \times \mathbf{1}$ column vector, all with entries that are either zero or simple unit fractions.

Observe that for $\mathbb{Z}^+ \times \mathbb{Z}^+$ lower triangular matrices, the end result of all arithmetic matrix operations, matrix inversion, application to infinite column vectors, etc., has appropriate n-th truncation equal to that obtained by first truncating all matrices, and then carrying out the operations. Hence these operations may be performed and the familiar identities of linear algebra continue to hold in this context.

Theorem. *Let* $\mathbf{u}$ *be the row vector* $\{u_k = 1/k : k \in \mathbb{Z}^+\}$, *where* $\mathbb{Z}^+$ *denotes the positive integers, and let* $\mathbf{M}$ *be the matrix with entries* $\{m_{ij} = 1/(i-j+1)$ if $j \le i$, $m_{ij} = 0$ if $j > i : i, j \in \mathbb{Z}^+\}$. *Let* $\mathbf{v}$ *be the column vector* $\{v_n = 1/(n+1) : n \in \mathbb{Z}^+\}$. *Then the product* $\mathbf{u}(\mathbf{M}^{-1}\mathbf{v})$ *exists (as a convergent series), and is equal to Euler's constant,*

$$\gamma = \lim_{m \to \infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \ln(m) \right) \approx 0.5772156649\ldots.$$

Explicitly,

$$\gamma = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \vdots \end{bmatrix} \right)$$

Proof: Substituting $t = 1 - e^{-x}$ in the standard definite integral

$$\gamma = \int_o^\infty \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x}\, dx \quad \text{yields} \quad \gamma = \int_0^1 \left(1 + \frac{t}{\ln(1-t)} \right) \frac{dt}{t}.$$

Write

$$\frac{t}{\ln(1-t)} = -\sum_{k=0}^{\infty} c_k t^k,$$

where the coefficients c_k are obtained from the formal division of power series. Since

$$\frac{\ln(1-t)}{t} = -\sum_{k=1}^{\infty} \frac{t^{k-1}}{k} \quad \text{on} \quad (-1,+1),$$

$\sum_{k=0}^{\infty} c_k t^k$ converges on some interval around 0, and $1 = (\sum_{k=1}^{\infty} t^{k-1}/k)(\sum_{k=0}^{\infty} c_k t^k)$ on this interval. Since $c_0 = \lim_{t\to 0} - t/\ln(1-t) = 1$, this is equivalent to the system of linear equations

$$\begin{aligned}
-\frac{1}{2} &= c_1 \\
-\frac{1}{3} &= \frac{c_1}{2} + c_2 \\
-\frac{1}{4} &= \frac{c_1}{3} + \frac{c_2}{2} + c_3 \\
-\frac{1}{5} &= \frac{c_1}{4} + \frac{c_2}{3} + \frac{c_3}{2} + c_4 \\
&\vdots
\end{aligned}$$

This system has the matrix form:

$$\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{3} \\ -\frac{1}{4} \\ -\frac{1}{5} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \end{bmatrix}$$

Thus we have $-\mathbf{v} = \mathbf{Mc}$, where $\mathbf{c}$ denotes the column vector $\{c_k : k \in \mathbb{Z}^+\}$. Eliminating the constant terms between two successive equations where c_{k-1} and c_k have unit coefficients, we have

$$c_k = \frac{1}{k+1} \sum_{j=1}^{k-1} \frac{j}{(k-j)(k-j+1)} c_j.$$

Consequently, by induction, $c_1 < 0, c_2 < 0, \cdots, c_{k-1} < 0$ implies $c_k < 0$, and again by induction $-1/(k+1) < c_k < 0$ $(k > 1)$. The latter inequality assures the convergence of $\sum_{k=1}^{\infty} c_k/k$. Therefore, using Abel's Theorem, we have

$$\gamma = -\lim_{x\to 1-} \int_0^x \left(\sum_{k=1}^{\infty} c_k t^k \right) \frac{dt}{t} = -\lim_{x\to 1-} \sum_{k=1}^{\infty} \frac{c_k}{k} x^k = -\sum_{k=1}^{\infty} \frac{c_k}{k} = -\mathbf{uc}.$$

Then $\mathbf{c} = (\mathbf{M}^{-1}\mathbf{M})\mathbf{c} = \mathbf{M}^{-1}(\mathbf{Mc}) = -\mathbf{M}^{-1}\mathbf{v}$. Using $\gamma = -\mathbf{uc}$, we obtain the result $\gamma = \mathbf{u}(\mathbf{M}^{-1}\mathbf{v})$.

We note that $\mathbf{M}$ is an unbounded operator on l_2. For example, using the Euclidean norm the sequence of unit vectors defined by

$$\mathbf{w}_m = \left\{ \{\mathbf{w}_m\}_n = \frac{1}{\sqrt{m}} \quad \text{for } n \le m, \quad \text{and } = 0 \text{ for } n > m \right\}$$

transforms into the sequence $\mathbf{M}\mathbf{w}_m$, which diverges because

$$\|\mathbf{M}\mathbf{w}_m\|^2 = \frac{1}{m} \sum_{n=1}^{m} \left(\sum_{k=1}^{n} \frac{1}{k} \right)^2$$

grows faster than $\ln(m!)/m \approx \ln m$, as m increases.

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More on a Mean Value Theorem Converse

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In a recent MONTHLY article Tong and Braza considered two possible versions of a converse to the Mean Value Theorem [**2**]. For $c \in (a, b)$, a continuous function f on $[a, b]$ that is differentiable on (a, b) satisfies the

1. Weak Form at c if $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$, and the
2. Strong Form at c if $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$ with $c \in (\alpha, \beta)$.

In [**2**] the authors give a function that fails the Weak Form (and so fails both forms) at all values in a countable closed set. Borwein and Wang provided a function that fails the Weak Form on a residual set (one whose complement is of first category) that is of Lebesgue measure zero [**1**].

We show that a differentiable function can fail the Weak Form on a set that is both residual and of relative measure arbitrarily close to 1 while the Strong Form must hold on some subset of positive Lebesgue measure. In the rest of this Note *measure* means Lebesgue measure, denoted by λ.

We consider $[a, b] = [0,1]$ and let Z be any measurable set in $[0, 1]$ with $\lambda(Z) < 1$. Let $E \subset [0, 1] \setminus Z$ be an F_σ set with $\lambda(E) = \lambda([0, 1] \setminus Z) > 0$ and E having density 1 at each $x \in E(\lim_{\epsilon \to 0} \lambda(E \cap (x - \epsilon, x + \epsilon))(2\epsilon)^{-1} = 1)$. Let g be an approximately continuous function (at each x the restriction of g to some subset with density 1 at x is continuous at x) such that:

$$\begin{aligned} &1.\ 0 < g(x) \le 1 \quad \text{for } x \in E, \quad \text{and} \\ &2.\ g(x) = 0 \quad \text{for } x \notin E. \end{aligned} \tag{1}$$

A construction of such functions can be found in Zahorski [**3**]. Since g is bounded