Change of Variables in Multiple Integrals

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Dedicated to the memory of Professor Clyde Klipple, who taught me real variables by the R. L. Moore method at Texas A & M in 1944.

1. Let $y = \varphi(x)$ be a differentiable mapping of the interval $S = [c, d]$. Denote by T the interval $[a, b]$ with $\varphi(c) = a$, $\varphi(d) = b$. Let f be a continuous function of y. The change of variable formula says that

$$\int_S f(\varphi(x)) \frac{d\varphi}{dx}\, dx = \int_T f(y)\, dy. \tag{1.1}$$

The usual proof uses the fundamental theorem of calculus. Denote by g an anti-derivative of f:

$$f = \frac{d}{dy} g. \tag{1.2}$$

According to the fundamental theorem of calculus,

$$\int_T f(y)\, dy = g(b) - g(a), \tag{1.3}$$

where a and b are the endpoints of the interval T. On the other hand, by the chain rule the derivative of the composite $g \circ \varphi$ is given by

$$\frac{d}{dx} g(\varphi(x)) = \frac{dg}{dy}\frac{d\varphi}{dx}.$$

Using (1.2) we see that the x derivative of $g \circ \varphi$ is the integrand on the left in (1.1); therefore by the fundamental theorem of calculus,

$$\int_S f(\varphi(x)) \frac{d\varphi}{dx}\, dx = g(\varphi(d)) - g(\varphi(c)), \tag{1.4}$$

where c and d are the endpoints of the interval S. Since $\varphi(c) = a$ and $\varphi(d) = b$, the right sides of (1.3) and (1.4) are the same; this completes the proof of (1.1).

The usual proof of the change of variable formula in several dimensions uses the approximation of integrals by finite sums; see for instance [7]. The purpose of this note is to show how to use the fundamental theorem of calculus to prove the change of variable formula for functions of any number of variables. Then, as a surprising byproduct, we obtain a proof of the Brouwer fixed point theorem. In the last section we compare our proof with other known analytic proofs of the fixed point theorem.

I thank Daniel Velleman for suggesting a substantial simplification of the argument.

2. In this section we study mappings $\varphi(x) = y$ of n-dimensional x space into n-dimensional y space. We impose two assumptions:

i) φ is once differentiable.
ii) φ is the identity outside some sphere, say the unit sphere:

$$\varphi(x) = x \quad \text{for } |x| \geq 1.$$

Change of variable theorem:. *Let f be a continuous function of compact support. Then*

$$\int f(\varphi(x))J(x)\,dx = \int f(y)\,dy, \tag{2.1}$$

where J is the Jacobian determinant of the mapping φ:

$$J(x) = \det\frac{\partial\varphi_j}{\partial x_i}; \tag{2.2}$$

here φ_j is the j^{th} component of φ.

We prove this for functions f that are once differentiable and for mappings φ that are twice differentiable; since functions and mappings can be approximated by differentiable ones, this suffices. The approximation can be accomplished by mollification, that is, by convolving each component of φ with a smooth, nonnegative, spherically symmetric function m with small support whose integral equals 1. As the support of m shrinks to zero, $m * \varphi$ and its first derivatives tend to those of φ. If φ is the identity, so is $m * \varphi$.

Define

$$g(y_1, y_2, \dots, y_n) = \int_{-\infty}^{y_1} f(z, y_2, \dots, y_n)\,dz. \tag{2.3}$$

Clearly, $\frac{\partial g}{\partial y_1} = f$. Since f is once differentiable, so is g. Since f has compact support, we can choose c so large that f is zero outside the c-cube

$$|y_i| \le c, \quad i = 1, 2, \dots, n.$$

It follows from (2.3) that $g(y_1, \dots, y_n) = 0$ when $|y_j| \ge c$ for any $j \ne 1$, and when $y_1 \le -c$.

Take $c \ge 1$; then the c-cube contains the unit ball. Since φ is the identity outside the unit ball, $f(\varphi(x))$ is zero outside the c-cube in x-space. So in the integrals in (2.1) we may restrict integration to the c-cube.

In the left side of (2.1), express f as the partial derivative of g:

$$\int \frac{\partial g}{\partial y_1}(\varphi(x))J(x)\,dx. \tag{2.4}$$

We denote by D the gradient with respect to x; the columns of the Jacobian matrix $\partial\varphi/\partial x$ are $D\varphi_1, \dots, D\varphi_n$.

Observation:. *The integrand in* (2.4) *can be written as the following determinant*:

$$\det(Dg(\varphi), D\varphi_2, \dots, D\varphi_n). \tag{2.5}$$

Proof: By the chain rule

$$Dg(\varphi) = \sum_{j=1}^{n} (\partial_{y_j} g) D\varphi_j. \tag{2.6}$$

We set this into the first column in (2.5). Formula (2.6) expresses $Dg(\varphi)$ as a linear combination of the vectors $D\varphi_1, D\varphi_2, \dots, D\varphi_n$; the last $n - 1$ of these vectors are the last $n - 1$ columns of the matrix in (2.5), and therefore these can be subtracted from $Dg(\varphi)$ without altering the value of the determinant (2.5). This leaves us with

$\det((\partial_{y_1} g(\varphi))D\varphi_1, D\varphi_2, \ldots, D\varphi_n)$; factoring out the scalar $(\partial_{y_1})g(\varphi)$ gives $(\partial_{y_1})g(\varphi)J$, the integrand in (2.4). ■

The next step is to expand the determinant (2.5) according to the first column; we obtain

$$M_1 \partial_{x_1} g(\varphi) + \cdots + M_n \partial_{x_n} g(\varphi), \tag{2.7}$$

where $M_1, \ldots, M_n$ are the cofactors of the first column of the Jacobian matrix. Setting (2.7) into the integrand in (2.4) we get

$$\int \left(M_1 \partial_{x_1} g(\varphi) + \cdots + M_n \partial_{x_n} g(\varphi) \right) dx. \tag{2.8}$$

Since φ is twice differentiable, we can integrate each term by parts over the c-cube and obtain

$$-\int g(\varphi)\left(\partial_{x_1} M_1 + \cdots + \partial_{x_n} M_n \right) dx + \textit{boundary terms}. \tag{2.9}$$

We use now the following classical identity:

$$\partial_{x_1} M_1 + \cdots + \partial_{x_n} M_n \equiv 0. \tag{2.10}$$

We sketch a proof: We can write the left side of (2.10) symbolically as

$$\det(D, D\varphi_2, \ldots, D\varphi_n). \tag{2.11}$$

For $n = 2$ we have

$$\det(D, D\varphi_2) = \partial_1 \partial_2 \varphi_2 - \partial_2 \partial_1 \varphi_2 = 0.$$

For $n > 2$ we note that the cofactors M_j are multilinear functions of the φ_j. Using the product rule of differentiation, we write (2.11), again symbolically, as

$$\sum_{2 \le k \le n} \det(D, D\varphi_2, \ldots, D\varphi_n)_k, \tag{2.12}$$

where the subscript k means that the differential operator D in the first column acts only on the k^{th} column. We leave it to the reader to verify that each of the determinants in the sum (2.12) is zero.

The identity (2.10) shows that the n-fold integral in (2.9) is zero.

We turn now to the boundary term in (2.9). Since $g(\varphi(x)) = g(x)$ on the boundary of the c-cube, the only nonzero boundary term is from the side $x_1 = c$; since $M_1 = 1$ when $\varphi(x) \equiv x$, that boundary term is

$$\int g(c, x_2, \ldots, x_n)\, dx_2 \cdots dx_n. \tag{2.13}$$

Using the definition (2.3) of g in (2.13) gives

$$\int \int_0^c f(z, x_2, \ldots, x_n)\, dz\, dx_2 \cdots dx_n,$$

which is the right side of equation (2.1). This completes the proof of the change of variables formula.

3. In our proof of the change of variables formula, we assumed neither that φ is one-to-one, nor that it is onto. We claim:

A mapping φ having properties i) and ii) of the change of variables theorem maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Suppose some point y_0 were not the image of any x. Since φ is the identity outside the unit ball, y_0 would lie inside the unit ball. Since φ maps $|x| \leq 1$ into a closed set, it would follow that some ball B_0 centered at y_0 would be free of images of φ. Now take any function f supported in the ball B_0, whose integral is nonzero:

$$\int f\,dy \neq 0. \tag{3.1}$$

By the change of variable formula

$$\int f(\varphi(x))J\,dx = \int f\,dy \neq 0. \tag{3.2}$$

Since the range of φ avoids B_0, and since the support of f lies in B_0, the integrand on the left in (3.2) is identically zero; then so is the integral. This contradicts (3.1), and so the claim is established.

Intermediate Value Theorem:. *Let φ be a continuous map of the unit ball in $\mathbb{R}^n$ into $\mathbb{R}^n$ that is the identity on the boundary:*

$$\varphi(x) = x \quad \text{for } |x| = 1.$$

Then the image of φ covers every point in the unit ball.

Proof: Extend φ to be the identity outside the unit ball. Then approximate the extended map by differentiable maps, each the identity outside the unit ball. According to our claim, each of these maps covers the unit ball. By compactness, so does their limit. ■

The following well-known argument shows how to deduce the Brouwer fixed point theorem from the intermediate value theorem.

Let ψ be a continuous mapping of the unit ball into the unit ball; we claim that it leaves a point fixed. If not then for every x there is a ray from $\psi(x)$ through x. This ray pierces the unit ball at a point that we denote by $\varphi(x)$. Clearly, φ is a continuous mapping; it is the identity for x on the unit sphere and maps the unit ball into the unit sphere. This contradicts the intermediate value theorem. ■

4. The Brouwer fixed point theorem has many analytical proofs. How do they compare with the present one? Hadamard [**3**] employed the identity (2.10) about the Jacobian matrix; so did Dunford-Schwartz [**2**, pp. 467–470].

Samelson [**6**] used Stokes' theorem to give an extremely short proof of the Brouwer fixed point theorem. This proof was rediscovered by Kannai [**5**]. According to Laurent Schwartz, as related by Haim Brézis, such a proof was current in Paris in the thirties.

Báez-Duarte [**1**] proved formula (2.1) using exterior forms and Stokes' theorem and deduced from it the intermediate value theorem. My deduction is the same as Báez-Duarte's.

The integration of exterior forms over chains presupposes the change of variable formula for multiple integrals. It is amusing that the change of variables formula alone implies Brouwer's theorem.

In conclusion we call attention to Erhardt Heinz's beautiful analytic treatment of the Brouwer degree of a mapping.

REFERENCES

1. L. Báez-Duarte, Brouwer's Fixed-Point Theorem and a Generalization of the Formula for Change of Variables in Multiple Integrals, *J. Math. Anal. Appl.* **177** (1993) 412–414.
2. N. Dunford and J. Schwartz, *Linear Operators, Part I*, Wiley-Interscience, New York, 1958.
3. J. Hadamard, Sur quelques applications de l'indice de Kronecker, pp. 437–477 in J. Tannery, *Introduction à la théorie des fonctions d'une variable*, vol. 2, Paris, 1910.
4. E. Heinz, An Elementary Analytic Theory of the Degree of Mapping n-Dimensional Space, *J. Math. Mech.* **8** (1959) 231–248.
5. Y. Kannai, An elementary proof of the no retraction theorem, *Amer. Math. Monthly* **88** (1981) 264–268.
6. H. Samelson, On the Brouwer fixed point theorem, *Portugal. Math.* **22** (1963) 189–191.
7. J. Schwartz, The formula for change in variables in a multiple integral, *Amer. Math. Monthly* **61** (1954) 81–85.

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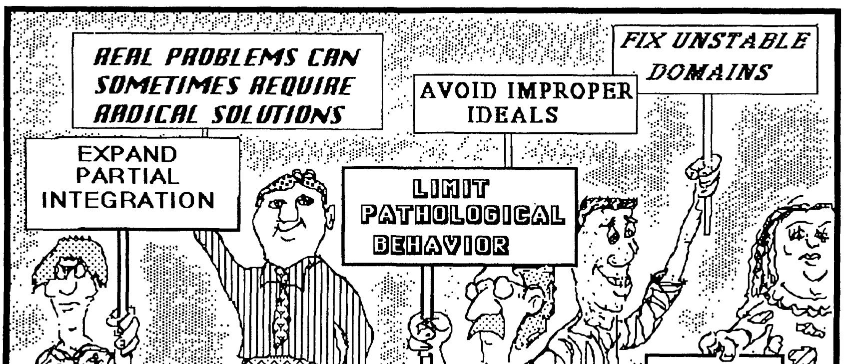

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