UNSOLVED PROBLEMS

Edited by **Richard Nowakowski**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Nowakowski, Department of Mathematics & Statistics & Computing Science, Dalhousie University, Halifax NS, Canada B3H 3J5, rjn@mscs.dal.ca

The Cayley Addition Table of Z_n

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Few mathematical objects could be considered more simple than the Cayley addition table of $\mathbf{Z}_n$ but we show that even these simple objects have some interesting yet unproved properties.

A *transversal* of an $n \times n$ matrix is a collection of n cells, no two of which are in the same row or column. A transversal of a matrix is a *latin* transversal if no two of its cells contain the same element.

Conjecture 1. For any odd n and any $k \in \{1, \dots, n\}$, any $k \times k$ submatrix of the Cayley addition table of $\mathbf{Z}_n$ contains a latin transversal.

Conjecture 2. For any even n and any $k \in \{1, \dots, n\}$, and any $k \times k$ submatrix of the Cayley addition table of $\mathbf{Z}_n$ contains a latin transversal provided the submatrix is not a subgroup of even order or a translate of such a subgroup.

Perhaps a stronger version of Conjecture 1 might be true.

Conjecture 3. Let A be the Cayley table of any Abelian group of odd order n, and let $k \in \{1, \dots, n\}$. Then any $k \times k$ submatrix of A contains a latin transversal.

Conjecture 3 implies a fundamental property of finite fields of odd characteristic: Let **GF[q]** be a finite field of odd characteristic containing q elements and let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$, $k \leq q$, be two subsets of **GF[q]**. Then there is a permutation $\pi \in S_k$ such that the sums $a_i + b_{\pi(i)}$, in (**GF[q]**), are pairwise distinct.

So far the only result that supports these conjectures (and the next) is the following theorem of Noga Alon. The special case $k = n$ for Conjectures 1, 2, and 3 was proved in [**2**].

Theorem 1 (Alon). *Let p be a prime, suppose $k < p$, let $(a_1, a_2, \dots, a_k)$ be a sequence of not necessarily distinct members of the finite field $\mathbf{Z}_p$, and let B be a subset* *of cardinality k of $\mathbf{Z}_p$. Then there is a numbering $(b_1, b_2, \ldots, b_k)$ of the elements of B such that the sums $a_i + b_i$ (in $\mathbf{Z}_p$) are pairwise distinct.*

Consider the infinitely long periodic sequence of integers modulo n. For example, modulo 6 the sequence is: 012345012345012345.... Now take k copies of this sequence ($k < n$) and start them at any position (i.e., truncate them anywhere).

For example, taking the sequence modulo 6 and $k = 3$,

012345012345012345...;
012345012345012345...;
012345012345012345...

all start at the same position and

012345012345012345...;
234501234501234501...;
501234501234501234....

all start at different positions. A $k \times k$ *frame* (matrix) consists of k consecutive columns taken from such an arrangement. For the second example all frames look like:

abc
cef
gab

This is the *template* of our k sequences.

Conjecture 4. Take k copies of the infinitely long periodic sequence modulo n (k < n) and start them anywhere (i.e., truncate them anywhere). Then the template has a latin transversal.

Note that Alon has proved this conjecture when n is a prime. The following examples show why k cannot equal n:

$n = k = 4$		$n = k = 5$
0123		01234
0123	and	01234
0123		01234
1230		01234
		12340

REFERENCES

1. N. Alon, *Additive Latin Transversals*, preprint.
2. M. Hall, Jr., A combinatorial problem on abelian groups, *Proc. Amer. Math. Soc.* **3** (1952) 584–587.

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