PROBLEMS AND SOLUTIONS

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before November 30, 1999; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10739. *Proposed by Oscar Ciaurri, Logroño, Spain.* Suppose that $f\colon [0,1] \to \mathbb{R}$ has a continuous second derivative with $f''(x) > 0$ on $(0,1)$, and suppose that $f(0) = 0$. Choose $a \in (0,1)$ such that $f'(a) < f(1)$. Show that there is a unique $b \in (a,1)$ such that $f'(a) = f(b)/b$.

10740. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, IL.* A connected bipartite simple graph has a unique bipartition, meaning a partition of the vertices into two independent sets. Let **G** be the set of such graphs that have no isomorphism that interchanges the two sets of the bipartition. Is there a criterion that for each $G \in \mathbf{G}$ selects a well-defined set of the bipartition?

10741. *Proposed by Tim Keller, Fair Oaks, CA.* Is there an even base b for which there exist square integers of the form $dddd_b$? By $dddd_b$, we mean the four-digit number in base b all of whose digits are d. For odd b we have the examples $1111_7 = 20^2$ and $4444_7 = 40^2$.

10742. *Proposed by Emre Alkan, University of Wisconsin, Madison, WI.* Let us say that a finite group G has the *maximal property* if, for any prime p that divides $|G|$, G has a maximal subgroup H such that $p|H|$ divides $|G|$.
(a) Show that every finite solvable group has the maximal property.
(b) Show that there are infinitely many finite groups with the maximal property that are not solvable.
(c) Show that there are infinitely many finite groups without the maximal property that are not solvable.

10743. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-La-Neuve, Belgium.* Let $p \geq 5$ be prime, and let n be an integer such that $(p+1)/2 \leq n \leq p-2$. Let $R = \sum(-1)^i\binom{n}{i}$, where the sum is taken over the quadratic residues i modulo p, and let $N = \sum(-1)^j\binom{n}{j}$, where the sum is taken over the quadratic nonresidues j modulo p. Prove that exactly one of R and N is divisible by p.