10744. *Proposed by Peter Lindqvist, Norwegian University of Science and Technology, Trondheim, Norway, and Jaak Peetre, University of Lund, Lund, Sweden.* Fix $p > 0$, and define functions $S(x)$, $C(x)$, and $T(x)$ for sufficiently small x by

$$x = \int_0^{S(x)} \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_{C(x)}^1 \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_0^{T(x)} \frac{dt}{(1+t^p)^{2/p}}.$$

Show that $S(x)^p + C(x)^p = 1$ and that $T(x) = S(x)/C(x)$. The case $p = 2$ yields the familiar trigonometric formulas.

10745. *Proposed by M. J. Pelling, London, England.* For $n \geq 1$, let $f(n)$ be the number of solutions (r, s, t) in positive integers to the Diophantine equation $rst = n(r+s+t)$.
(a) Prove that $f(n) = O(n^{1/2+\delta})$ for every $\delta > 0$.
(b)* Prove that $f(n) = O(n^{\delta})$ for every $\delta > 0$.

SOLUTIONS

Using the Walls to Find the Center

10386 [1994, 474]. *Proposed by Jordan Tabov, Bulgarian Academy of Sciences, Sofia, Bulgaria.* Let a tetrahedron with vertices A_1, A_2, A_3, A_4 have altitudes that meet in a point H. For any point P, let P_1, P_2, P_3, and P_4 be the feet of the perpendiculars from P to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Prove that there exist constants a_1, a_2, a_3, and a_4 such that one has

$$a_1\overrightarrow{PP_1} + a_2\overrightarrow{PP_2} + a_3\overrightarrow{PP_3} + a_4\overrightarrow{PP_4} = \overrightarrow{PH}$$

for every point P.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. More generally, let H and P be any two points in the space of the given tetrahedron and let P_1, P_2, P_3, P_4 be the feet of the lines through P parallel to HA_1, HA_2, HA_3, HA_4 in the faces of the tetrahedron opposite A_1, A_2, A_3, A_4, respectively. Then there exist constants a_1, a_2, a_3, a_4, independent of P, such that

$$a_1\overrightarrow{PP_1} + a_2\overrightarrow{PP_2} + a_3\overrightarrow{PP_3} + a_4\overrightarrow{PP_4} = \overrightarrow{PH}.$$

Let $\mathbf{V}$ denote the vector from an origin outside the space of the given tetrahedron to any point V in the space of the tetrahedron. Then H and P have the representations (barycentric coordinates)

$$\mathbf{H} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + x_3\mathbf{A}_3 + x_4\mathbf{A}_4 \quad (x_1 + x_2 + x_3 + x_4 = 1),$$

$$\mathbf{P} = u_1\mathbf{A}_1 + u_2\mathbf{A}_2 + u_3\mathbf{A}_3 + u_4\mathbf{A}_4 \quad (u_1 + u_2 + u_3 + u_4 = 1).$$

Since P_1 has the representation $\mathbf{P}_1 = r_2\mathbf{A}_2 + r_3\mathbf{A}_3 + r_4\mathbf{A}_4$, where $r_2 + r_3 + r_4 = 1$, we must have

$$r_2\mathbf{A}_2 + r_3\mathbf{A}_3 + r_4\mathbf{A}_4 - \mathbf{P} = \lambda_1(\mathbf{H} - \mathbf{A}_1).$$

Since $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ are independent vectors, we get $\lambda_1 = u_1/(1-x_1)$, so that $\overrightarrow{PP_1} = (\mathbf{P}_1 - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_1)u_1/(1-x_1)$. Similarly,

$$(\mathbf{P}_i - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_i)\frac{u_i}{1-x_i} \quad \text{for } i = 1, 2, 3, 4.$$

Choosing $a_i = 1 - x_i$, we obtain

$$\sum a_i(\mathbf{P}_i - \mathbf{P}) = \sum u_i(\mathbf{H} - \mathbf{A}_i) = \mathbf{H} - \mathbf{P} = \overrightarrow{PH}.$$

This proof generalizes to give an analogous result for n-dimensional simplices.

Solved also by J. Anglesio (France), R. J. Chapman (U. K.), M. Golomb, K. Hanes, N. Komanda, O. P. Lossers (The Netherlands), and the proposer.