

10745



M. J. Pelling

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**10744.** Proposed by Peter Lindqvist, Norwegian University of Science and Technology, Trondheim, Norway, and Jaak Peetre, University of Lund, Lund, Sweden. Fix  $p > 0$ , and define functions  $S(x)$ ,  $C(x)$ , and  $T(x)$  for sufficiently small  $x$  by

$$x = \int_0^{S(x)} \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_{C(x)}^1 \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_0^{T(x)} \frac{dt}{(1+t^p)^{2/p}}.$$

Show that  $S(x)^p + C(x)^p = 1$  and that  $T(x) = S(x)/C(x)$ . The case  $p = 2$  yields the familiar trigonometric formulas.

**10745.** Proposed by M. J. Pelling, London, England. For  $n \geq 1$ , let  $f(n)$  be the number of solutions  $(r, s, t)$  in positive integers to the Diophantine equation  $rst = n(r + s + t)$ .

(a) Prove that  $f(n) = O(n^{1/2+\delta})$  for every  $\delta > 0$ .

(b)\* Prove that  $f(n) = O(n^\delta)$  for every  $\delta > 0$ .

## SOLUTIONS

### Using the Walls to Find the Center

**10386** [1994, 474]. Proposed by Jordan Tabov, Bulgarian Academy of Sciences, Sofia, Bulgaria. Let a tetrahedron with vertices  $A_1, A_2, A_3, A_4$  have altitudes that meet in a point  $H$ . For any point  $P$ , let  $P_1, P_2, P_3$ , and  $P_4$  be the feet of the perpendiculars from  $P$  to the faces  $A_2A_3A_4, A_3A_4A_1, A_4A_1A_2$ , and  $A_1A_2A_3$ , respectively. Prove that there exist constants  $a_1, a_2, a_3$ , and  $a_4$  such that one has

$$a_1 \overrightarrow{PP_1} + a_2 \overrightarrow{PP_2} + a_3 \overrightarrow{PP_3} + a_4 \overrightarrow{PP_4} = \overrightarrow{PH}$$

for every point  $P$ .

*Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.* More generally, let  $H$  and  $P$  be any two points in the space of the given tetrahedron and let  $P_1, P_2, P_3, P_4$  be the feet of the lines through  $P$  parallel to  $HA_1, HA_2, HA_3, HA_4$  in the faces of the tetrahedron opposite  $A_1, A_2, A_3, A_4$ , respectively. Then there exist constants  $a_1, a_2, a_3, a_4$ , independent of  $P$ , such that

$$a_1 \overrightarrow{PP_1} + a_2 \overrightarrow{PP_2} + a_3 \overrightarrow{PP_3} + a_4 \overrightarrow{PP_4} = \overrightarrow{PH}.$$

Let  $\mathbf{V}$  denote the vector from an origin outside the space of the given tetrahedron to any point  $V$  in the space of the tetrahedron. Then  $H$  and  $P$  have the representations (barycentric coordinates)

$$\mathbf{H} = x_1 \mathbf{A}_1 + x_2 \mathbf{A}_2 + x_3 \mathbf{A}_3 + x_4 \mathbf{A}_4 \quad (x_1 + x_2 + x_3 + x_4 = 1),$$

$$\mathbf{P} = u_1 \mathbf{A}_1 + u_2 \mathbf{A}_2 + u_3 \mathbf{A}_3 + u_4 \mathbf{A}_4 \quad (u_1 + u_2 + u_3 + u_4 = 1).$$

Since  $P_1$  has the representation  $\mathbf{P}_1 = r_2 \mathbf{A}_2 + r_3 \mathbf{A}_3 + r_4 \mathbf{A}_4$ , where  $r_2 + r_3 + r_4 = 1$ , we must have

$$r_2 \mathbf{A}_2 + r_3 \mathbf{A}_3 + r_4 \mathbf{A}_4 - \mathbf{P} = \lambda_1 (\mathbf{H} - \mathbf{A}_1).$$

Since  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$  are independent vectors, we get  $\lambda_1 = u_1/(1-x_1)$ , so that  $\overrightarrow{PP_1} = (\mathbf{P}_1 - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_1)u_1/(1-x_1)$ . Similarly,

$$(\mathbf{P}_i - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_i) \frac{u_i}{1-x_i} \quad \text{for } i = 1, 2, 3, 4.$$

Choosing  $a_i = 1 - x_i$ , we obtain

$$\sum a_i (\mathbf{P}_i - \mathbf{P}) = \sum u_i (\mathbf{H} - \mathbf{A}_i) = \mathbf{H} - \mathbf{P} = \overrightarrow{PH}.$$

This proof generalizes to give an analogous result for  $n$ -dimensional simplices.

Solved also by J. Anglesio (France), R. J. Chapman (U. K.), M. Golomb, K. Hanes, N. Komanda, O. P. Lossers (The Netherlands), and the proposer.