A Reciprocal Summation Identity: 10490

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A Reciprocal Summation Identity

10490 [1995, 930]. *Proposed by Seung-Jin Bang, Ajou University, Suwon, Korea.* Show that

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{j}\right)=\sum_{k=1}^{n}\frac{1}{k^3}$$

for each positive integer n.

Solution I by Jeremy E. Dawson, Australian National University, Canberra, Australia. Since

$$\int_0^1\frac{1}{z}\int_0^z\frac{1}{y}\int_0^y x^{k-1}\,dx\,dy\,dz=\frac{1}{k^3},$$

we have $\sum_{k=1}^{n}1/k^3=\int_0^1 z^{-1}\int_0^z y^{-1}\int_0^y(1-x^n)/(1-x)\,dx\,dy\,dz$. Letting $w=1-x$, $v=1-y$, and $u=1-z$, we can rewrite the latter integral as

$$\int_0^1\frac{1}{1-u}\int_u^1\frac{1}{1-v}\int_v^1\frac{1-(1-w)^n}{w}dw\,dv\,du.$$

Now use $\left(1-(1-w)^n\right)/w=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}w^{k-1}$ and interchange the order of summation and integration. For the resulting multiple integrals, we use

$$\frac{1}{1-s}\int_s^1 t^{m-1}dt=\frac{1-s^m}{m(1-s)}=\frac{1}{m}\sum_{l=1}^{m}s^{l-1}$$

twice to obtain

$$\int_0^1\frac{1}{1-u}\int_u^1\frac{1}{1-v}\int_v^1 w^{k-1}dw\,dv\,du=\int_0^1\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{j}\sum_{i=1}^{j}u^{i-1}\right)du=\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i}.$$

Thus

$$\sum_{k=1}^{n}\frac{1}{k^3}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{1}{i}.$$

Solution II by A. N. 't Woord, University of Technology, Eindhoven, The Netherlands. We prove first that if $b_n=\sum_{k=1}^{n}a_k\binom{n}{k}$ for $n\geq 1$, then

$$(1)\quad \sum_{k=1}^{n}\frac{b_k}{k}=\sum_{k=1}^{n}\frac{a_k}{k}\binom{n}{k},\qquad\text{and}\qquad (2)\quad a_n=\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}b_k\ \text{ for } n\geq 1.$$

(1) follows by induction on n: it is immediate for $n=1$; and for $n>1$ we have

$$\sum_{k=1}^{n}\frac{a_k}{k}\binom{n}{k}=\sum_{k=1}^{n}\frac{a_k}{k}\binom{n-1}{k}+\sum_{k=1}^{n}\frac{a_k}{k}\binom{n-1}{k-1}=\sum_{k=1}^{n-1}\frac{b_k}{k}+\sum_{k=1}^{n}\frac{a_k}{n}\binom{n}{k}=\sum_{k=1}^{n}\frac{b_k}{k}.$$

For (2), we use $\binom{n}{k}\binom{k}{i}=\binom{n}{i}\binom{n-i}{n-k}$ and $\sum_{k=i}^{n}(-1)^{n-k}\binom{n-i}{n-k}=\delta_{in}$ to obtain

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}b_k=\sum_{k=1}^{n}\sum_{i=1}^{k}(-1)^{n-k}\binom{n}{k}\binom{k}{i}a_i=\sum_{i=1}^{n}a_i\binom{n}{i}\sum_{k=i}^{n}(-1)^{n-k}\binom{n-i}{n-k}=a_n.$$

We now begin with the identity $1=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}$. Applying (1) twice yields

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}.$$

Applying (2) now yields

$$\frac{(-1)^{n-1}}{n^2} = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{j} \frac{1}{i}.$$

Dividing by $(-1)^{n-1}$ and applying (1) once more yields the desired identity.

Solution III by O. P. Lossers, University of Technology, Eindhoven, The Netherlands. Let

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{k} \frac{1}{i} (1-x)^i.$$

We prove that $f(x) - f(0) = -\sum_{k=1}^{n} x^k/k^3$. Since $f(1) = 0$, this yields the desired identity $f(0) = \sum_{k=1}^{n} 1/k^3$.

Let Δ be the transformation taking $p(x)$ to $xp'(x)$. Direct computation shows that $\Delta^3(-\sum_{k=1}^{n} x^k/k^3) = -\sum_{k=1}^{n} x^k$. Since Δ is injective on the space of polynomials with constant term 0, it suffices to show that $\Delta^3\big(f(x) - f(0)\big) = -\sum_{k=1}^{n} x^k$ as well.

The linearity of Δ yields $\Delta^3\big(f(x) - f(0)\big) = \Delta^3\big(f(x)\big)$. Now we compute directly

$$\begin{aligned}
\Delta^3\big(f(x)\big) &= \Delta^2\left(\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1}{k} \sum_{j=1}^{k} \frac{1-(1-x)^j}{j}\right) \\
&= \Delta\left(\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1-(1-x)^k}{k}\right) \\
&= x \sum_{k=1}^{n} \binom{n}{k} (-1)^k (1-x)^{k-1} = x \frac{x^n - 1}{1-x} = -\sum_{k=1}^{n} x^k.
\end{aligned}$$

Solution IV by Víctor Hernández, Universidad Nacional de Educación a Distancia, Madrid, Spain. The desired identity is the case $m = 3$ of the more general formula

$$\sum_{1 \le i_1 \le \cdots \le i_m \le n} \frac{(-1)^{i_m - 1}}{i_1 \cdots i_m} \binom{n}{i_m} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

Consider two probabilistic processes for partitioning the unit interval $[0, 1]$ into $m + 1$ subintervals. Let $X_0 = 1$ and $Y_0 = 0$. Choose X_{i+1} uniformly at random in $[0, X_i]$, and choose Y_{i+1} uniformly at random in $[Y_i, 1]$. By symmetry, X_m and $1 - Y_m$ have the same distribution. We compute

$$\mathbf{E}(X_m^{k-1}) = \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \cdots \int_0^{x_{m-1}} x_m^{k-1}\, dx_m = \frac{1}{k^m} \qquad \text{and}$$

$$\mathbf{E}(Y_m^{k-1}) = \int_0^1 \frac{dy_1}{1-y_1} \int_{y_1}^1 \frac{dy_2}{1-y_2} \cdots \int_{y_{m-1}}^1 y_m^{k-1}\, dy_m = \frac{1}{k} \sum_{i_{m-1}=1}^{k} \frac{1}{i_{m-1}} \cdots \sum_{i_1=1}^{i_2} \frac{1}{i_1}.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^m} &= \mathbf{E}\left(\sum_{k=1}^{n} X_m^{k-1}\right) = \mathbf{E}\left(\frac{1 - X_m^n}{1 - X_m}\right) = \mathbf{E}\left(\frac{1 - (1 - Y_m)^n}{Y_m}\right) \\
&= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \mathbf{E}\left(Y_m^{k-1}\right) = \sum_{1 \le i_1 \le \cdots \le i_m \le n} \frac{(-1)^{i_m - 1}}{i_1 \cdots i_m} \binom{n}{i_m},
\end{aligned}$$

where we have set $i_m = k$.

Editorial comment. Other solvers used generating functions, inductive arguments, and various identities and transformations. Joe Howard and Heinz-Jürgen Seiffert independently observed that the result follows fairly quickly from a result proved by the proposer in his solution to Problem 4427, *School Science and Mathematics* 95 (1995) 221.

Solved also by E. S. Andersen & M. E. Larsen (Denmark), J. C. Binz (Switzerland), P. Bracken, D. Bradley (Canada), E. Braune (Austria), D. Callan, R. J. Chapman (U. K.), D. A. Darling, V. Dwivedi (India), E. Hertz, M. Hoffman, J. Howard, A. Kaplan (France), A. A. Kelzon (Russia), R. A. Kopas, O. Krafft & M. Schaefer (Germany), J. H. Lindsey II, J. Lorch, P. McCartney, D. K. Nester, A. Pechtl (Germany), F. Qi (China), E. Schmeichel, L. Scribani (South Africa), H.-J. Seiffert (Germany), A. Sinefakopoulos (Greece), I. Sofair, R. Sprugnoli (Italy), A. Stadler (Switzerland), A. Stenger, J. Van hamme (Belgium), M. Vowe (Switzerland), Z. Wu, GCHQ Problems Group (U. K.), WMC Problems group, and the proposer.

A Limit of Periods

10603 [1997, 567]. *Proposed by Yury J. Ionin and Robin R. Lewis, Central Michigan University, Mt. Pleasant, MI.* Let a, b, and k be positive integers, and let $P_k(a, b)$ be the period of the sequence $\{a^n \bmod b^k\}_{n=1}^{\infty}$. Find $\lim_{k\to\infty} P_{k+1}(a, b)/P_k(a, b)$.

Solution by the proposers. The limit equals the largest divisor of b that is relatively prime to a.

Suppose first that a and b are relatively prime. Fixing a and b, let $P_k = P_k(a, b)$. We have $a^{P_k} \equiv 1 \bmod b^k$, so $a^{P_k} = 1 + q_k b^k$ for some integer q_k. Note that P_k divides P_{k+1}. Thus $P_{k+1} = u_k P_k$, where u_k is the smallest positive integer u such that $a^{uP_k} \equiv 1 \bmod b^{k+1}$. Since

$$a^{uP_k} = (1 + q_k b^k)^u \equiv (1 + uq_k b^k) \bmod b^{k+1},$$

we have $u_k = b/d_k$, where $d_k = \gcd(q_k, b)$. Thus $P_{k+1} = bP_k/d_k$.

If $k \geq 2$, then the equalities $a^{P_{k+1}} = 1 + q_{k+1}b^{k+1} = (1 + q_k b^k)^{b/d_k}$ imply that $q_{k+1} = (q_k/d_k) + t_k q_k^2 b$, where t_k is an integer. Therefore, if p is a common prime divisor of b and q_k, then p occurs in the prime factorization of q_{k+1} with an exponent smaller than its exponent in the prime factorization of q_k. If p is a prime divisor of b that does not divide q_k, then p also does not divide q_{k+1}. If k is sufficiently large, this implies that q_k and b are relatively prime, so $d_k = 1$ and $P_{k+1} = bP_k$. It follows that the desired limit is b.

Suppose now that a and b are not relatively prime. Let r be the largest divisor of b that is relatively prime to a, and let $b = rs$. Now $P_k(a, b)$ is a period of the sequence $\{a^n \bmod r^k\}_{n=1}^{\infty}$, and thus $P_k(a, r)$ divides $P_k(a, b)$.

On the other hand, consider the expression $a^{P_k(a,r)+m} - a^m$. This is divisible by r^k, and for large m it is divisible by s^k. Since $\gcd(r, s) = 1$, it must therefore be divisible by b^k, and we discover that $P_k(a, r)$ is a period of the sequence $\{a^n \bmod b^k\}_{n=1}^{\infty}$. Hence $P_k(a, b)$ divides $P_k(a, r)$. We conclude that $P_k(a, b) = P_k(a, r)$. Since a and r are relatively prime, the claim follows from the previous case.

Solved also by D. A. Callan, R. J. Chapman (U. K.), J. H. Lindsey II, A. N. 't Woord (The Netherlands), GCHQ Problems Group, and NCCU Problems Group.

Avoiding the Identity

10606 [1997, 664]. *Proposed by Thomas Zaslavsky, Binghamton University, Binghamton, NY.* Given a positive integer m, show that there is a positive integer n such that, for every group G of order at least n, it is possible to choose m elements $g_1, g_2, \ldots, g_m$ so that no product of the form $g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \cdots g_{i_k}^{\pm 1}$ with $1 \leq k \leq m$ and distinct subscripts $i_1, i_2, \ldots, i_k$ in $\{1, 2, \ldots, m\}$ equals the identity.

Solution by Stephen M. Gagola, Jr., Kent State University, Kent, OH. Let a *signed product* be a product of distinct group elements or their inverses in a specified order. A set S is *admissible* if the identity is not expressible as a signed product of elements of S. We prove