that every group of order at least $f(m) = \lceil 2^{m-1}(m-1)!\sqrt{e}\,\rceil$ has an admissible subset of size m.

For $S \subseteq G$, let S^* be the set of signed products of elements of S. Let $g(m) = f(m+1)-2$. When $|S| = m$, we claim that $|S^*| \le g(m)$. A signed product is formed by choosing an ordered nonempty subset of S with exponents ± 1. Thus $|S^*| \le \sum_{k=1}^{m} \binom{m}{k} k! 2^k$. We can rewrite the bound as $2^m m! T_{m-1}(1/2)$, where $T_{m-1}(x) = \sum_{j=0}^{m-1} x^j/j!$ is the Maclaurin polynomial of degree $m-1$ for e^x. The next term in the series expansion of $2^m m! e^{1/2}$ contributes 1, while the remainder after that is at most 1. Thus $|S^*| \le f(m+1) - 2$.

Note that S^* and $G - S^*$ are closed under taking inverses. If a signed product equals the identity, then each of its elements can be expressed as a signed product of the other elements in the product. If S is admissible and x is a nonidentity element of $G - S^*$, it therefore follows that $S \cup \{x\}$ is also admissible. Thus S can be enlarged until $G - S^*$ contains only the identity element.

We now use induction on m to prove the claim that every group of order at least $f(m)$ has an admissible subset of size m. When $m = 1$, every nontrivial group has a nonidentity element, and this forms an admissible set of size 1. This agrees with $f(1) = 2$. When $m > 1$, the monotonicity of f and the induction hypothesis imply that every group of order at least $f(m)$ has an admissible subset S of size $m-1$. Since $g(m-1) = f(m) - 2$, there is a nonidentity element in $G - S^*$, and S can be enlarged by one element.

Solved also by P. J. Anderson (Canada), R. Barbara (France), D. Beckwith, G. L. Body (U. K.), R. J. Chapman (U. K.), J. H. Lindsey II, S. C. Locke, J. Merickel, K. A. Ross, M. C. Slattery, GCHQ Problems Group (U. K.), NCCU Problems Group, NSA Problems Group, and the proposer.

Some Sums Require Care

10638 [1998, 69]. *Proposed by Brian Conolly, Cambridge, U. K.* For $0 \le \lambda \le 1$ and $m \ge 0$, let $S_m(\lambda) = \sum_{n\ge1} e^{-\lambda n}(\lambda n)^{n-m}/n!$. Show that $S_0(\lambda) = \lambda/(1-\lambda)$, $S_1(\lambda) = 1$, $S_2(\lambda) = 1/\lambda - 1/2$, and $S_3(\lambda) = 1/\lambda^2 - 3/(4\lambda) + 1/6$.

Solution I by Allen Stenger, Tustin, CA. Let

$$T_m(\lambda) = \lambda^m S_m(\lambda) = \sum_{n\ge1} \frac{(\lambda e^{-\lambda})^n n^{n-m}}{n!}.$$

By Stirling's formula the summand is asymptotic to $(2\pi)^{-1/2} n^{-m-1/2} (\lambda e^{1-\lambda})^n$. Thus, this sum converges absolutely for $|\lambda e^{1-\lambda}| < 1$. Therefore it represents a continuous function on $[0, 1)$. Furthermore, if $m \ge 1$, it converges uniformly for $|\lambda e^{1-\lambda}| \le 1$ and therefore is continuous on $[0, 1]$. For $m = 0$ it diverges at $\lambda = 1$.

First consider the case $m = 1$. We want to show that $\sum_{n\ge1} (\lambda e^{-\lambda})^n n^{n-1}/n! = \lambda$. Euler showed that this holds for $0 \le \lambda \le 1$. It can be derived by applying the Lagrange inversion formula to $\lambda e^{-\lambda}$ (G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Volume 1, Springer, 1972, Part 3, Exercise 209).

The formulas for other values of m can be derived from the case $m = 1$ by integration or differentiation. Observe that $T_m(0) = 0$ and that by uniform convergence

$$\begin{aligned}\frac{d}{d\lambda} T_m(\lambda) &= \frac{d}{d\lambda} \sum_{n\ge1} \frac{\lambda^n e^{-n\lambda} n^{n-m}}{n!} \\ &= \sum_{n\ge1} \frac{\lambda^{n-1} e^{-n\lambda} n^{n-(m-1)}}{n!} - \sum_{n\ge1} \frac{\lambda^n e^{-n\lambda} n^{n-(m-1)}}{n!} = \frac{1-\lambda}{\lambda} T_{m-1}(\lambda),\end{aligned}$$

for $\lambda \in [0, 1)$. Thus

$$T_0(\lambda) = \frac{\lambda}{1-\lambda}\frac{d}{d\lambda}T_1(\lambda) = \frac{\lambda}{1-\lambda},$$
$$T_2(\lambda) = \int_0^\lambda \frac{1-z}{z}T_1(z)\,dz = \lambda - \frac{1}{2}\lambda^2, \quad \text{and}$$
$$T_3(\lambda) = \int_0^\lambda \frac{1-z}{z}T_2(z)\,dz = \lambda - \frac{3}{4}\lambda^2 + \frac{1}{6}\lambda^3$$

for $0 \le \lambda < 1$. Continuity extends the latter two formulas to $\lambda = 1$.

Solution II by Thomas Hermann, SDRC, Milford, OH. Replacing $e^{-\lambda n}$ by its Taylor series and rearranging the sum formally gives

$$S_m(\lambda) = \sum_{n=1}^{\infty}\frac{(\lambda n)^{n-m}}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{(\lambda n)^k}{k!} = \sum_{r=1}^{\infty}\frac{(-1)^r\lambda^{r-m}}{r!}\sum_{n=1}^{r}(-1)^n\binom{r}{n}n^{r-m}. \quad (*)$$

This reordering of the sum is justified as follows. By the uniform convergence proved in Solution I, the first sum in $(*)$ represents an analytic function on the domain $\{\lambda : |\lambda e^{1-\lambda}| < 1\}$. For sufficiently small $|\lambda|$, this sum converges absolutely. Thus, the reordering is valid for $|\lambda|$ small. Hence the second sum in $(*)$ is the Laurent series for the analytic function given by the first sum in some punctured neighborhood of 0. We show that this Laurent series converges for $0 < |\lambda| < 1$ if $m = 0$ and for $\lambda \neq 0$ if $m \ge 1$. Therefore these two analytic functions agree on the connected domain $\{\lambda : |\lambda e^{1-\lambda}| < 1 \text{ and } 0 < |\lambda| < 1\}$ and on its boundary if $m \ge 1$. Thus, the two sides of $(*)$ agree for $\lambda \in [0, 1)$, and if $m \ge 1$ for $\lambda \in [0, 1]$.

The inner sum on the right side of $(*)$ is found in I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 1994, equations 0.154 (3) and (4). It can also be derived by evaluating at 0 the $(r-m)$th derivative of $(e^t - 1)^r$ in two different ways. One obtains

$$\sum_{n=1}^{r}(-1)^n\binom{r}{n}n^{r-m} = \begin{cases} 0 & \text{if } 1 \le m < r; \\ (-1)^r r! & \text{if } m = 0. \end{cases}$$

Therefore

$$S_0(\lambda) = \sum_{r=1}^{\infty}\frac{(-\lambda)^r}{r!}(-1)^r r! = \frac{\lambda}{1-\lambda},$$

which holds for $|\lambda| < 1$, and for $m \ge 1$,

$$S_m(\lambda) = \sum_{r=1}^{m}\left(\sum_{n=1}^{r}(-1)^{r-n}\frac{n^{r-m}}{n!(r-n)!}\right)\left(\frac{1}{\lambda}\right)^{m-r},$$

which holds for all $\lambda \neq 0$.

Editorial comment. The proposer notes that the problem of showing $S_1(1) = 1$ was posed by P. J. Cameron in connection with a result of Rényi concerning random forests of rooted trees. In queuing theory, $S_1(\lambda)$ is the probability that a busy period will ever end in the process M/D/1 with traffic intensity λ. Since $S_1(\lambda) < 1$ for $\lambda > 1$, a busy period for that process has nonzero probability of continuing forever. When $0 < \lambda < 1$, the mean number of customers served during a busy period is $S_0(\lambda)/\lambda$.

Solved also by P. Bracken (Canada), D. Callan, R. J. Chapman (U. K.), G. L. Isaacs, J. H. Lindsey II, O. P. Lossers (The Netherlands), V. Lucic (Canada), R. Mortini (France), V. Schindler (Germany), H.-J. Seiffert (Germany), A. Sofo (Australia), A. Stadler (Switzerland), A. Tissier (France), T. V. Trif (Romania), J. Van hamme (Belgium), M. Vowe (Switzerland), WMC Problems Group, and the proposer.