An Extension of the Wallace-Simson Theorem: Projecting in Arbitrary Directions

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$x \le y$ the estimates

$$|B_n(f,x) - B_n(f,y)| \le \mathbf{E}|f(S_n(x)) - f(S_n(y))| \le L\mathbf{E}|S_n(x) - S_n(y)|^\alpha$$
$$= L\mathbf{E}\left|\frac{1}{n}\sum_{j=1}^{n}\chi_{[x,y)}(u_j)\right|^\alpha \le L\left(\mathbf{E}\left|\frac{1}{n}\sum_{j=1}^{n}\chi_{[x,y)}(u_j)\right|\right)^\alpha$$
$$\le L|x-y|^\alpha. \qquad \blacksquare$$

This proof is brief and elementary, but heavily uses the specific realization of the $S_n(x)$ through an empirical distribution function, thereby correlating the random variables $S_n(x)$ and $S_n(y)$ properly. This was also crucial for proving Lemma 2. The simple arguments leading to Theorem 1 did not rely on any specific realization.

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1. THE WALLACE-SIMSON LINE. The Wallace line has been a popular object of study for many geometers during the two past centuries. Let us start by recalling the theorem.

The Wallace-Simson Theorem. *Consider a triangle ABC. The locus of all those points P in its plane such that the orthogonal projections of P on the three sides of the triangle are collinear is the circumcircle of ABC. The line of the projections is called the Wallace-Simson line of P with respect to ABC.*

Poncelet attributed this theorem to Robert Simson (1687–1768), among others, since it had the flavor of the geometrical properties that Simson was interested in. Almost certainly it is actually due to Wallace, another Scottish mathematician of lesser importance, who discovered it in 1799.

The beauty of this theorem, which shows a somewhat unexpected and surprising relationship between a triangle and its circumcircle, attracted many geometers in the nineteenth century, among others Jakob Steiner, and led to the discovery of many beautiful properties. Among the most surprising are those connected with Feuerbach's circle, Steiner's deltoid, and Morley's triangle. A reader interested in making an excursion through this landscape can find a guide in the references at the end of this article. Particularly interesting for its numerous references to the older literature is F.G.-M.'s work [**7**, p. 329]. The initials F.G.-M correspond to Fr. Gabriel-Marie, who signed his published works this way.

It is rather simple to obtain a direct proof of the Wallace theorem. One has just to prove that, in Figure 1, one has $\angle BVU = \angle AVW$. Observing that quadrilateral

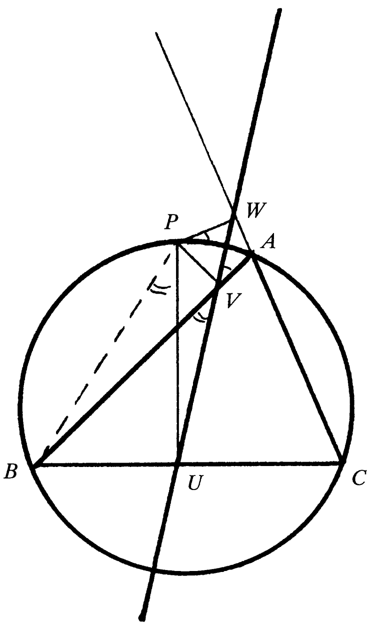

Figure 1

$PVUB$ is cyclic because of the right angles at U and V one gets $\angle BVU = \angle BPU = 90° - \angle PBU = 90° - \angle PBC$. In the same way $PVAW$ is also cyclic and therefore $\angle AVW = \angle APW = 90° - \angle PAW = 90° - (180° - \angle PAC) = \angle PAC - 90°$. But it is quite clear that $\angle PAC - 90° = 90° - \angle PBC$ since the angles at A and B are supplementary in cyclic quadrilateral $PBCA$, and so $\angle BVU = \angle AVW$, which shows that U, V, and W are collinear. ■

2. AN EXTENSION OF THE THEOREM. The goal of this Note is to present a generalization of the Wallace theorem along a strikingly simple direction that I have never seen explored. I first state the theorem and discuss its proof and then present some nice exercises that can be solved in a simple way.

Theorem. *Consider a triangle ABC. Let us call a, b, c, the corresponding sides opposite the vertices. We fix three projection directions α, β, γ, not all three equal, and such that α is not parallel to side a, β is not parallel to side b, and γ is not parallel to side c. Take an arbitrary point P in the plane of ABC and project it on a along α obtaining U, on b along β obtaining V, and on c along γ obtaining W as in Figure 2. Fix a real number k and an orientation in the plane in order to give a sign to the areas of the triangles we consider.*

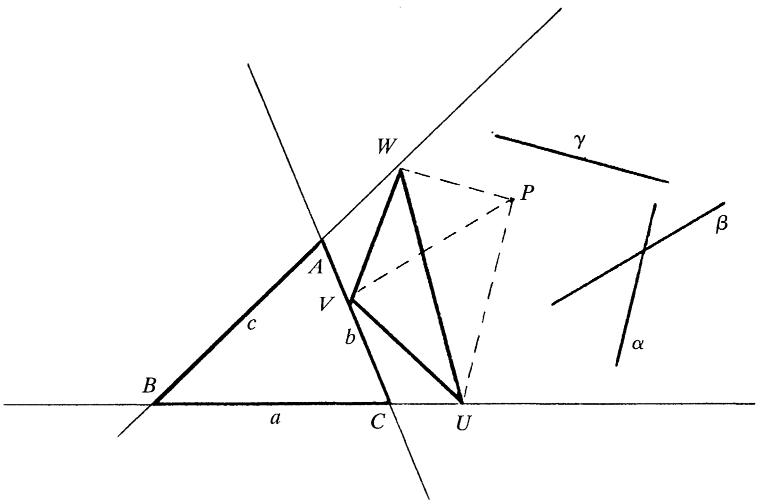

Figure 2

Then the locus of all points P such that the oriented triangle UVW has area k is a conic $\mathbf{C}(k)$. It is clear that $\mathbf{C}(0)$ always goes through the three vertices A, B, C (for example, with P at A, V and W are also at A). The conic $\mathbf{C}(k)$ can, of course, degenerate in various ways.

When k varies (with fixed α, β, γ), the family $\mathbf{C}(k)$ is always a family of conics with the same points at infinity. Furthermore, if one of them has a center, all others have the same center and are homothetical to each other (except the possibly degenerate elements of the family $\mathbf{C}(k)$), the homothecy center being the common center of all such conics. If none has a center, then they are all translations of the same parabola along the direction of its axis.

The construction, with straightedge and compass, of the common center (when it exists) and of the axes and asymptotes of the conics of the family $\mathbf{C}(k)$ is easily done once one knows A, B, C and the projection directions α, β, γ.

Proof: The theorem is one of those results whose only difficulty is arriving at its statement, since the easy analytical proof we develop could be left as an exercise to the reader.

Let us begin by fixing an arbitrary cartesian system of coordinates and an orientation to give a sign to the areas of the triangles. If the point P has coordinates (x, y) and we denote by α, β, γ three vectors that correspond to the given projection directions, it is clear that the points U, V, W have as coordinates, respectively,

$$(u_1(x,y), u_2(x,y)), (v_1(x,y), v_2(x,y)), (w_1(x,y), w_2(x,y))$$

where each of these functions is a linear function in x, y with coefficients that depend only on the parameters (already fixed) that determine $A, B, C, \alpha, \beta, \gamma$, in our coordinate system.

The area of the triangle UVW is given by half the value of the determinant of the matrix

$$\begin{pmatrix} u_1(x,y) & u_2(x,y) & 1 \\ v_1(x,y) & v_2(x,y) & 1 \\ w_1(x,y) & w_2(x,y) & 1 \end{pmatrix}.$$

Therefore the equation of the locus is of the form

$$mx^2 + ny^2 + 2pxy + 2qx + 2ry + s = 2k \tag{1}$$

where m, n, p, q, r, s depend only on the fixed entities $A, B, C, \alpha, \beta, \gamma$. This shows that $\mathbf{C}(k)$ is a conic and that, when k varies, $\mathbf{C}(k)$ is a family of conics whose points at infinity are the same, since they are determined by

$$mx^2 + ny^2 + 2pxy = 0. \tag{2}$$

The equation (2) cannot degenerate, i.e., m, n, p cannot be 0 simultaneously. In fact, in that case $\mathbf{C}(0)$ would have as equation $2qx + 2ry + s = 0$. If one of the coefficients q, r is not zero, then $\mathbf{C}(0)$ is a straight line, which is false, since A, B, C are points of $\mathbf{C}(0)$. If $q = r = 0$ then the equation of $\mathbf{C}(0)$ is $s = 0$. Therefore, if s is not 0, $\mathbf{C}(0)$ is empty, which is again false. If $s = 0$ then $\mathbf{C}(0) = \mathbb{R}^2$, which easily implies that all three directions α, β, γ, are the same; this was excluded from the beginning.

Therefore, $\mathbf{C}(k)$ is a family of conics with the same points at infinity, as announced in the statement of the theorem. If we take as the origin an arbitrary point O and as coordinate axes the two (always real) bisectors of the angles formed by the lines joining O with the points at infinity, then the equation (1) takes the form

$$mx^2 + ny^2 + 2qx + 2ry + s = 2k \tag{3}$$

(where m, n, q, r, s, of course, need not be the same as before).

If for some k the conic $\mathbf{C}(k)$ has a point Z as center and we take Z as the origin and as coordinate axes the axes of the conic (the two bisectors of the angle obtained by joining Z to the two points at infinity), then the equation (3) takes the form

$$mx^2 + ny^2 + s = 2k \tag{4}$$

with m, n, s depending only, as before, on $A, B, C, \alpha, \beta, \gamma$, which makes plain the simple structure of the family $\mathbf{C}(k)$. All, except for $s = 2k$, are concentric and homothetic with respect to Z. For $s = 2k$ the conic $\mathbf{C}(k)$ degenerates and becomes either a pair of real lines (which can coincide) or a pair of imaginary lines that intersect at Z.

If in equation (4) it happens, for example, that $m = 0$ then the equation of $\mathbf{C}(k)$ would be $ny^2 + s = 2k$, which is, for each k, a pair of parallel lines, real or complex according to the value of k; this pair of lines becomes one double real line if $s = 2k$. It is clear, in this case, that any point of such a double line is a center of symmetry of each $\mathbf{C}(k)$.

If no $\mathbf{C}(k)$ has a center, then in the equation

$$mx^2 + ny^2 + 2qx + 2ry + s = 2k$$

we have either $m = 0$ or $n = 0$. Assume $n = 0$. Then

$$mx^2 + 2qx + 2ry + s = 2k$$

and if we make the change $x = X - q/m$ (i.e., a translation of the y axis) we obtain

$$mX^2 + 2ry + t = 2k$$

with m, r, t depending only, as before, on $A, B, C, \alpha, \beta, \gamma$. The coefficient r cannot be null, since if $r = 0$ we would have $mX^2 + t = 2k$, and this is, for each k, a pair of parallel lines and thus has a center. Therefore, r is not null and so $\mathbf{C}(k)$ is, in this case, a family of parabolas which are all obtained as translations of one of them in the direction of its axis.

Construction, with compass and straightedge, of the center, axes and asymptotes of $\mathbf{C}(k)$.

Here we indicate a practical construction for the main elements of the conics $\mathbf{C}(k)$. According to the previous part of the theorem it is sufficient to find these elements for $\mathbf{C}(0)$, of which we already know three points A, B, C.

If two of the projection directions coincide, for example $\beta = \gamma$, it is easy to show in a direct way that $\mathbf{C}(0)$ is the line BC together with the line through A in the direction $\beta = \gamma$, independent of the direction α. It is also easy to show that coincidence of two of the projection directions is a necessary and sufficient condition for degeneration of the conic $\mathbf{C}(0)$ into two lines (one of them is the line containing one of the sides of the triangle and the other is the line through the opposite vertex in the same direction as the two coinciding ones). It is then quite clear that, if $\beta = \gamma$ does not coincide with the direction of the side BC, then $\mathbf{C}(k)$, with k different from 0, is a family of hyperbolas having these two lines as asymptotes that are homothetic to each other with respect to their common center, the intersection of those two lines. When $\beta = \gamma$ has the same direction as the side BC, each $\mathbf{C}(k)$ is a pair of parallel lines (that coincide for $k = 1/2$) and parallel to those that constitute $\mathbf{C}(0)$. Thus, we have determined the main elements of $\mathbf{C}(k)$ if $\mathbf{C}(0)$ degenerates.

If no pair of projection directions coincide, then $\mathbf{C}(0)$ is a non-degenerate conic for which we already know three points. Another three are easily determined in the following way:

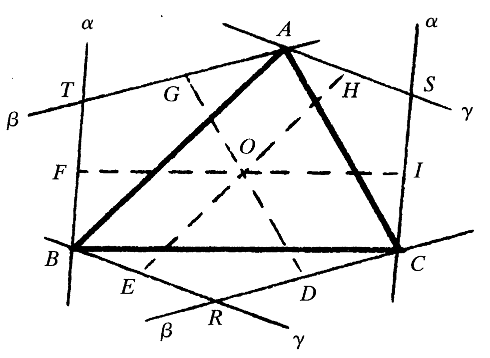

Figure 3

Through B we draw a line parallel to γ and through C a line parallel to β; this makes $V = C$ and $W = B$, and UVW degenerates; it is clear that the intersection R belongs to $\mathbf{C}(0)$. Through C we draw a line parallel to α and through A a line parallel to γ; similarly the intersection S belongs to $\mathbf{C}(0)$. Through A we draw a line parallel to β and through B a line parallel to α; the intersection T belongs to $\mathbf{C}(0)$.

We thus obtain a hexagon $ATBRCS$ whose vertices are in $\mathbf{C}(0)$ and such that its opposite sides are parallel. It is clear that the midpoints of the sides of a pair of opposite sides are on a diameter of $\mathbf{C}(0)$. We thus obtain the center of $\mathbf{C}(0)$ (when it does exist) as intersection of these three diameters, DG, EH, FI (see Figure 3). When the center does not exist we obtain at least the direction of the axis of the parabola $\mathbf{C}(0)$.

Assume for the moment that there is a center as indicated in Figure 3. Since, we already have three pairs of conjugate diameters, the construction, with compass and straightedge, of the common asymptotes and axes of all conics $\mathbf{C}(k)$ is a well known exercise.

For the determination of the vertex of the parabola $\mathbf{C}(0)$ when the center does not exist, we proceed as follows. Through A we draw a line perpendicular to the direction of the axis, which we already know. We determine its intersection A' with the parabola; the bisector line of AA' is the axis of $\mathbf{C}(0)$. The intersection of this line with the parabola gives us the vertex. The parabolas $\mathbf{C}(k)$ are the translations of $\mathbf{C}(0)$ in the direction of its axis.

3. SOME INTERESTING EXERCISES RELATED TO THE THEOREM. The theorem just proved suggests many interesting exercises, which can be solved easily with the methods we have used in the proof. Here are a few with some indications of the way to solve them. Fix a triangle ABC.

1. *Find necessary and sufficient conditions on the projection directions in order that* $\mathbf{C}(0)$ *and* $\mathbf{C}(k)$ *are circles. (Of course,* $\mathbf{C}(0)$ *will be the circumcircle and all* $\mathbf{C}(k)$ *will be concentric to it.)*

Hint: Consider the inscribed hexagon in $\mathbf{C}(0)$ *that we used in the construction of the main elements of* $\mathbf{C}(0)$. *The orthogonal projections to the sides (the Wallace Theorem) are not the only possibilities in order to get circles.*

2. *Determine necessary and sufficient conditions on the projection directions in order that the conics* $\mathbf{C}(k)$ *are equilateral hyperbolas.*

Hint: Remember that a necessary and sufficient condition for a conic that goes through three points A, B, C, *to be an equilateral hyberbola is that it also goes through the orthocenter of* ABC.

3. *For two different projection directions, determine a third projection so that the* $\mathbf{C}(k)$ *are equilateral hyperbolas.*

Hint: See the hint for Exercise 2.

4. *Given three directions* α, β, γ, *find, if possible, a point* F *such that the triangle* MNP *with vertices at the projections of* F *on the sides* a, b, c *in the directions* α, β, γ *has maximum or minimum area.*

Hint: Consider the equation of the conic $\mathbf{C}(k)$ *with respect to its axes.*

5. *Given a non-degenerate conic* $\mathbf{C}$ *and the triangle* ABC *inscribed in it, determine three directions* α, β, γ *such that* $\mathbf{C}$ *is the conic* $\mathbf{C}(0)$ *for* ABC *and the three given directions.*

Hint: Remember the hexagon inscribed in the conic that we used in the construction of the elements of $\mathbf{C}(0)$.

6. *Assume that triangle ABC has area S and that the radius of its circumscribed circle G is R. We draw a circle K concentric with G and with radius r. From a point P of K we draw its projections U, V, W on the sides of ABC. Determine, as a function of S, R, and r, the area of the triangle UVW.*

Hint: The same as in Exercise 4. Answer: $Area(UVW) = (S/4)(1 - r^2/R^2)$, *having selected the appropriate orientation so that the triangle UVW has positive area when* $r < R$.

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Another Short Proof of Ramanujan's Mod 5 Partition Congruence, and More

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We present another novel short proof of Ramanujan's partition congruence

$$p(5n + 4) \equiv 0 \pmod 5 \tag{1}$$

in addition to that presented by John L. Drost [**2**], and indeed prove rather more.

Ramanujan made the remarkable observation from a table of values of $p(n)$, the number of partitions of n, that $p(5n + 4)$ is divisible by 5. He observed and conjectured much more, and his conjectures turned out in the main to be correct. He gave a simple proof, based upon identities of Euler and Jacobi, of the conjecture (1), and his proof is essentially the one reproduced in Hardy and Wright [**3**] and referred to by Drost. Ramanujan's proof relies on manipulating power series, and considering coefficients modulo 5. It is my intention to give a proof of a