10750. *Proposed by Leonard Smiley, University of Alaska, Anchorage, AK.* For a positive integer m, express $\sum_{n=1}^{\infty}(n/\gcd(m,n))x^n$ as a rational function of x.

10751. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Let n be a positive integer, and let S_n be the set of all strings $a_1a_2\cdots a_n$ of positive integers satisfying $a_1 = 1$ and $a_{i+1} - a_i \in \{1, -1, -3, -5, \ldots\}$. For example, $S_5 = \{12345, 12343, 12341, 12323, 12321, 12123, 12121\}$. Find $|S_n|$.

10752. *Proposed by Gh. Costovici, Technical University "Gh. Asachi", Iasi, Romania.* For $n \in \mathbb{N}$, let a_n and b_n be complex numbers, with each $b_n \neq 0$. Let $s_n = a_1 + a_2 + \cdots + a_n$, and let $t_n = (1 - b_1/b_{n+1})\,a_1 + (1 - b_2/b_{n+1})\,a_2 + \cdots + (1 - b_n/b_{n+1})\,a_n$.
(a) Prove that if $\lim_{n\to\infty} b_{n+1}/b_n = 1$ and $\sum_{n=1}^{\infty} |s_n - t_n|^q$ converges for some $q \in (0, 1]$, then $\sum_{n=1}^{\infty} a_n$ converges.
(b) Prove that if $\sum_{n=1}^{\infty} |b_{n+1}/b_n - 1|^r$ and $\sum_{n=1}^{\infty} |s_n - t_n|^{r/(r-1)}$ converge for some $r \in (1, \infty)$, then $\sum_{n=1}^{\infty} a_n$ converges.

SOLUTIONS

A Zeta Function over a Recurrent Sequence

10486 [1995, 841]. *Proposed by Joseph H. Silverman, Brown University, Providence, RI.* Let $a, b > 0$ and $\alpha > 1$ be real numbers, and define $Z(s) = \sum_{n\in\mathbb{Z}} \left(a\alpha^n + b\alpha^{-n}\right)^{-s}$ for complex numbers s with positive real part.
(a) Prove that $Z(s)$ has a meromorphic continuation to all of $\mathbb{C}$.
(b) Find the poles of $Z(s)$.
(c) Find the residues of $Z(s)$ at its poles.

Solution I by David Bradley, University of Maine, Orono, ME. Let σ be the real part of s. Write

$$Z(s) = (a+b)^{-s} + \sum_{n=1}^{\infty}\left(a\alpha^n + b\alpha^{-n}\right)^{-s} + \sum_{n=1}^{\infty}\left(b\alpha^n + a\alpha^{-n}\right)^{-s}. \tag{1}$$

Without loss of generality, assume that $0 < a \le b$. We first consider the case $|\alpha| > \sqrt{b/a}$. We then have the two binomial expansions

$$\left(a\alpha^n + b\alpha^{-n}\right)^{-s} = \frac{a^{-s}\alpha^{-ns}}{\left(1 + ba^{-1}\alpha^{-2n}\right)^s} = a^{-s}\alpha^{-ns}\left(\sum_{k=0}^{m-1}\binom{-s}{k}\frac{b^k}{a^k}\alpha^{-2nk} + E_{m,n}(s)\right) \tag{2}$$

and

$$\left(b\alpha^n + a\alpha^{-n}\right)^{-s} = \frac{b^{-s}\alpha^{-ns}}{\left(1 + ab^{-1}\alpha^{-2n}\right)^s} = b^{-s}\alpha^{-ns}\left(\sum_{k=0}^{m-1}\binom{-s}{k}\frac{a^k}{b^k}\alpha^{-2nk} + F_{m,n}(s)\right), \tag{3}$$

where m is a fixed positive integer and $E_{m,n}(s) = O(\alpha^{-2mn})$ and $F_{m,n}(s) = O(\alpha^{-2mn})$. Since $|\alpha| > \sqrt{b/a}$, it follows from (1)–(3) that

$$\begin{aligned} Z(s) &= (a+b)^{-s} + \sum_{k=0}^{m-1}\binom{-s}{k}\left(\frac{b^k}{a^{s+k}} + \frac{a^k}{b^{s+k}}\right)\sum_{n=1}^{\infty}\alpha^{-n(s+2k)} + O\left(\sum_{n=1}^{\infty}\alpha^{-n(\sigma+2m)}\right) \\ &= (a+b)^{-s} + \sum_{k=0}^{m-1}\binom{-s}{k}\frac{a^{-s-k}b^k + b^{-s-k}a^k}{\alpha^{s+2k} - 1} + O\left(\sum_{n=1}^{\infty}\alpha^{-n(\sigma+2m)}\right). \end{aligned} \tag{4}$$

Since $E_{m,n}(s)$ and $F_{m,n}(s)$ are analytic for $\sigma > -2m$, it follows by analytic continuation that (4) is valid for $\sigma > -2m$. Since m is an arbitrary positive integer, we conclude that $Z(s)$ has a meromorphic continuation to the entire complex plane.