



## A Matrix of Inequalities: 10599

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*The American Mathematical Monthly*, Vol. 106, No. 7. (Aug. - Sep., 1999), p. 688.

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*The American Mathematical Monthly* is currently published by Mathematical Association of America.

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functions. For even  $s$  (C. B. Ling, On summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 5 (1974) 551–561) and for all positive integral  $s$  (I. J. Zucker, The summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 10 (1979) 192–206),  $Z(s)$  can be evaluated in terms of elliptic functions. In particular, for  $s = 1$ ,  $0 < k < 1$ , and  $z = K(\sqrt{1-k^2})/K(k)$ , we have  $\sum_{n=-\infty}^{\infty} \cosh^{-1}(n\pi z) = (\sum_{n=-\infty}^{\infty} e^{-\pi n^2 z})^2 = (2/\pi)K(k)$ , where  $K(k)$  is the complete elliptic integral of the first kind (see B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, 1991, p. 102 and p. 138).

Solved also by D. Cantor, R. J. Chapman (U. K.), R. Holzinger, and the proposer.

### A Matrix of Inequalities

**10599** [1997, 566]. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let  $x_1, \dots, x_m$  and  $y_1, \dots, y_n$  be nonnegative numbers and let  $(a_{ij})$  be an  $m \times n$  matrix of nonnegative numbers with at least one nonzero entry in each row. Suppose that the inequality  $\sum_{h=1}^m a_{hj}x_h \leq \sum_{k=1}^n a_{ik}y_k$  holds whenever  $a_{ij} > 0$ . Show that  $\sum_{i=1}^m x_i \leq \sum_{j=1}^n y_j$ .

*Solution by Frank Jelen and Eberhard Triesch, Der Rheinisch-Westfälischen Technischen Hochschule, Aachen, Germany.* Let  $A$  be the specified matrix, with columns  $c_1, \dots, c_n$ . Let  $x = (x_1, \dots, x_m)^T$  and  $y = (y_1, \dots, y_n)^T$ , and let  $\mathbf{1}_k$  denote the column vector of length  $k$  with entries equal to 1.

Define  $b = (b_1, \dots, b_m)^T$  by  $b_i = \max\{c_j^T x : a_{ij} > 0\}$ ; this is well-defined since each row contains a positive entry. Consider the linear programs

$$\text{minimize } \mathbf{1}_n^T z \quad \text{subject to } Az \geq b \text{ and } z \geq 0 \quad (1)$$

and

$$\text{maximize } b^T w \quad \text{subject to } A^T w \leq \mathbf{1}_n \text{ and } w \geq 0. \quad (2)$$

These linear programs are duals of each other, and (1) has the feasible solution  $z = y$ . It thus suffices to show that there exists a feasible solution  $u$  of (2) with  $b^T u \geq \mathbf{1}_m^T x$ , since the Duality Theorem then yields  $\mathbf{1}_n^T y \geq b^T u \geq \mathbf{1}_m^T x$ .

Consider the nonnegative vector  $u = (u_1, \dots, u_m)^T$  defined by  $u_i = x_i/b_i$  if  $b_i > 0$  and  $u_i = 0$  otherwise. Clearly  $b^T u = \mathbf{1}_m^T x$ .

For  $1 \leq j \leq n$ , define  $I_j = \{i : a_{ij} > 0 \text{ and } x_i > 0\}$ . For  $i \in I_j$ , we have  $b_i \geq c_j^T x > 0$ . Feasibility of  $u$  now follows from

$$c_j^T u = \sum_{i=1}^m a_{ij}u_i = \sum_{i \in I_j} a_{ij} \frac{x_i}{b_i} \leq \frac{1}{c_j^T x} \sum_{i \in I_j} a_{ij}x_i = 1.$$

Solved also by the proposer.

### A Complex Determinant

**10601** [1997, 566]. *Proposed by Wen-Xiu Ma, Universität-GH Paderborn, Paderborn, Germany.* Let  $n > 1$  be an integer and let  $a_1, a_2, \dots, a_n$  be complex numbers. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{2n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{2n-1} \\ \vdots & & \ddots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{2n-1} \\ 0 & 1 & 2a_1 & \cdots & (2n-1)a_1^{2n-2} \\ 0 & 1 & 2a_2 & \cdots & (2n-1)a_2^{2n-2} \\ \vdots & & \ddots & & \vdots \\ 0 & 1 & 2a_n & \cdots & (2n-1)a_n^{2n-2} \end{vmatrix} = (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (a_i - a_j)^4.$$