A Matrix of Inequalities: 10599

Fred Galvin; Frank Jelen; Eberhard Triesch

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functions. For even s (C. B. Ling, On summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 5 (1974) 551–561) and for all positive integral s (I. J. Zucker, The summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 10 (1979) 192–206), $Z(s)$ can be evaluated in terms of elliptic functions. In particular, for $s = 1$, $0 < k < 1$, and $z = K(\sqrt{1-k^2})/K(k)$, we have $\sum_{n=-\infty}^{\infty} \cosh^{-1}(n\pi z) = (\sum_{n=-\infty}^{\infty} e^{-\pi n^2 z})^2 = (2/\pi)K(k)$, where $K(k)$ is the complete elliptic integral of the first kind (see B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, 1991, p. 102 and p. 138).

Solved also by D. Cantor, R. J. Chapman (U. K.), R. Holzsager, and the proposer.

A Matrix of Inequalities

10599 [1997, 566]. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be nonnegative numbers and let (a_{ij}) be an $m \times n$ matrix of nonnegative numbers with at least one nonzero entry in each row. Suppose that the inequality $\sum_{h=1}^{m} a_{hj}x_h \le \sum_{k=1}^{n} a_{ik}y_k$ holds whenever $a_{ij} > 0$. Show that $\sum_{i=1}^{m} x_i \le \sum_{j=1}^{n} y_j$.

Solution by Frank Jelen and Eberhard Triesch, Der Rheinisch-Westfälischen Technischen Hochschule, Aachen, Germany. Let A be the specified matrix, with columns $c_1, \dots, c_n$. Let $x = (x_1, \dots, x_m)^T$ and $y = (y_1, \dots, y_n)^T$, and let $\mathbf{1}_k$ denote the column vector of length k with entries equal to 1.

Define $b = (b_1, \dots, b_m)^T$ by $b_i = \max\{c_j^T x\colon a_{ij} > 0\}$; this is well-defined since each row contains a positive entry. Consider the linear programs

$$\text{minimize} \quad \mathbf{1}_n^T z \qquad \text{subject to} \quad Az \ge b \text{ and } z \ge 0 \tag{1}$$

and

$$\text{maximize} \quad b^T w \qquad \text{subject to} \quad A^T w \le \mathbf{1}_n \text{ and } w \ge 0. \tag{2}$$

These linear programs are duals of each other, and (1) has the feasible solution $z = y$. It thus suffices to show that there exists a feasible solution u of (2) with $b^T u \ge \mathbf{1}_m^T x$, since the Duality Theorem then yields $\mathbf{1}_n^T y \ge b^T u \ge \mathbf{1}_m^T x$.

Consider the nonnegative vector $u = (u_1, \dots, u_m)^T$ defined by $u_i = x_i/b_i$ if $b_i > 0$ and $u_i = 0$ otherwise. Clearly $b^T u = \mathbf{1}_m^T x$.

For $1 \le j \le n$, define $I_j = \{i : a_{ij} > 0 \text{ and } x_i > 0\}$. For $i \in I_j$, we have $b_i \ge c_j^T x > 0$. Feasibility of u now follows from

$$c_j^T u = \sum_{i=1}^{m} a_{ij}u_i = \sum_{i \in I_j} a_{ij}\frac{x_i}{b_i} \le \frac{1}{c_j^T x}\sum_{i \in I_j} a_{ij}x_i = 1.$$

Solved also by the proposer.

A Complex Determinant

10601 [1997, 566]. *Proposed by Wen-Xiu Ma, Universität-GH Paderborn, Paderborn, Germany.* Let $n > 1$ be an integer and let $a_1, a_2, \dots, a_n$ be complex numbers. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{2n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{2n-1} \\ \vdots & & & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{2n-1} \\ 0 & 1 & 2a_1 & \cdots & (2n-1)a_1^{2n-2} \\ 0 & 1 & 2a_2 & \cdots & (2n-1)a_2^{2n-2} \\ \vdots & & & \ddots & \vdots \\ 0 & 1 & 2a_n & \cdots & (2n-1)a_n^{2n-2} \end{vmatrix} = (-1)^{n(n-1)/2} \prod_{1 \le i < j \le n} (a_i - a_j)^4.$$