Editorial comment. From $(*)$ we infer that $a(l, 1, n)$ is the total number of arrangements with at most l ascending runs that can be formed from subsets of $[n]$.

Solved also by R. J. Chapman (U. K.), Q. Darwish (Oman), H.-J. Seiffert (Germany), and the proposer.

A Variation on Additive Bases

10610 [1997, 664]. *Proposed by Richard Hall, University of Portsmouth, Portsmouth, England.* Given a positive integer m, let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two not necessarily distinct elements of S. For example, $C(3) = 8$ with $S = \{1, 3, 4\}$.
(**a**) Show that, for all $\epsilon > 0$, $1/4 < C(m)/m^2 < 1/2 + \epsilon$ for all sufficiently large m.
(**b**)* Improve the asymptotic bounds in part (**a**).

Solution to (**a**) *by the National Security Agency Problems Group, Fort Meade, MD.* Let $[n]l$ denote the first n positive multiples of l. When m is even, with $m = 2t$, let $S = [t-1]1 \cup [t+1]t$. Since S has size m and represents all positive integers up to $(t+1)t+t$, we have $C(m) \geq t^2 + 2t$. Thus $C(m)/m^2 \geq (t^2+2t)/(2t)^2 > 1/4$.

When m is odd, with $m = 2t+1$, let $S = [t]1 + [t+1](t+1)$. Since S has size m and represents all positive integers up to $(t+1)^2+t+1$, we have $C(m) \geq (t+1)(t+2)$. Thus $C(m)/m^2 \geq (t+1)(t+2)/(2t+1)^2 > 1/4$.

A set of size m represents at most $2m + \binom{m}{2}$ integers. Hence $C(m)/m^2 \leq 1/2 + 3/(2m) < 1/2 + \epsilon$ for $m > 3/(2\epsilon)$.

Solution to (**b**) *by the GCHQ Problems Group, Cheltenham, UK.* We show that $9/32 < C(m)/m^2 < 4/9 + \epsilon$ for all sufficiently large m.

For the lower bound, we construct a set that represents many integers by spreading the summands apart more quickly than in (**a**). Write m as $16i + j$, where $-7 \leq j \leq 8$, and let $A = [1, 3i]$, $B = [2, 7i+j]3i$, $C = (7i+j)3i + [1, 3i](3i+1)$, and $D = (7i+j)6i + 6i + [0, 3i]$, where $[x, y] = \{n \in \mathbb{Z}\colon x \leq n \leq y\}$. Let $S = A \cup B \cup C \cup D$.

From A and $A + A$ we get $[1, 6i]$, from $A + B$ we get $[6i+1, (7i+j)3i+3i]$, and from C and $A + C$ we get $[(7i+j)3i+3i+1, 30i^2+3i(j+2)]$.

For $r \in [1, 4i+j]$ and $s \in [1, 3i]$, we have $(3i+r-s+1)3i \in B$ and $(7i+j)3i + s(3i+1) \in C$, and the sum of these integers is $(10i+r+j+1)3i+s$. Thus $B+C$ contains $[(10i+2+j)3i+1, (14i+2j+1)3i+3i]$, which equals $[30i^2+3i(j+2)+1, (7i+j)6i+6i]$. Furthermore, $D \cup (A+D) \cup (B+D) = [(7i+j)6i+6i, (7i+j)9i+9i]$, and $C + D = [(7i+j)9i+9i+1, (7i+j)9i+3i(3i+1)+9i] = [(7i+j)9i+9i+1, 72i^2+i(12+9j)]$.

Since $|A \cup B \cup C \cup D| = m$, for large enough i we have

$$\frac{C(m)}{m^2} \geq \frac{72i^2 + i(12+9j)}{256i^2 + 32ij + j^2} = \frac{9}{32}\frac{8i^2 + i(j+4/3)}{8i^2 + ij + j^2/32} > \frac{9}{32}.$$

To prove that $C(m)/m^2 < 4/9 + \epsilon$, we show that some of the $m + \binom{m}{2}$ pairs must be "wasted". This happens in two ways. First, the sum may be too big, as happens for any pair of numbers that both exceed $C(m)/2$. Second, note that $r - s = t - u$ if and only if $r + u = t + s$. Thus we obtain a wasted pair for each instance of identical differences.

Consider a set S that represents everything from 1 to μm^2, for some $\mu > 1/4$. We may assume that $S \subseteq [1, \mu m^2]$. Let $am = \left|S \cap [1, \mu m^2/2]\right|$. All pairs from the $(1-a)m$ numbers above $\mu m^2/2$ are wasted. The smaller pairs have differences between 1 and $\mu m^2/2 - 1$, yielding wastage when $am + \binom{am}{2} > \mu m^2/2 - 1$.

Let $a \succeq b$ mean that $a > b - \epsilon$ for large enough m. Letting wm^2 be the number of wasted pairs, we have $w \succeq \max(0, (a^2 - \mu)/2) + (1-a)^2/2$. Letting $f(a)$ denote this lower bound, we have $f'(a) = -(1-a)$ for $a^2 < \mu$ and $f'(a) = 2a - 1$ for $a^2 > \mu$. The first quantity is negative and the second positive, since $\mu > 1/4$. Thus w is minimized at

$a^2 = \mu$, and hence $w \succeq (1-\sqrt{\mu})^2/2$. Excluding the wasted sums yields $\mu \preceq 1/2 - w$, and so $2\mu \preceq 1-(1-\sqrt{\mu})^2 = 2\sqrt{\mu}-\mu$. Thus $\sqrt{\mu} \preceq 2/3$ and $\mu \preceq 4/9$.

Editorial comment. John Lindsey proved that $C(m)/m^2 < .499785077 + \epsilon$. Kevin Ford proved that $C(m)/m^2 < .48832 + \epsilon$ and conjectured that the number on the right can be replaced by $\pi/8 < .393$.

Solved also by R. J. Chapman, K. Ford, J. H. Lindsey II, and the proposer.

An Identity Involving Rooted Trees

10615 [1997, 767]. *Proposed by Joaquin Gómez Rey, Alcorcón, Madrid, Spain.* For n a positive integer, evaluate

$$\sum (k_1 + k_2 + \cdots + k_n)! \prod_{i=1}^{n} \frac{i^{(i-1)k_i}}{(k_i!)(i!)^{k_i}}$$

where the summation runs over all n-tuples $(k_1, k_2, \ldots, k_n)$ of nonnegative integers such that $k_1 + 2k_2 + \cdots + nk_n = n$.

Solution I by Anchorage Math Solutions Group, University of Alaska, Anchorage, AK. We show that the given expression a_n equals $n^n/n!$.

Let f, g, h be exponential generating functions with coefficients f_n, g_n, h_n, respectively. When $f(g(x)) = h(x)$, expanding the composition and collecting terms appropriately yields

$$h(x) = \sum_{n=0}^{\infty} n! \left(\sum_{l=1}^{n} f_l \cdot \sum \frac{\prod_{j=1}^{l} g_{\lambda_j}}{\prod_i (k_i!)(i!)^{k_i}} \right) \frac{x^n}{n!},$$

where the innermost sum is over partitions λ of n that have k_i parts of size i for each i and l parts all told. When $f_n = n!$ and $g_n = n^{n-1}$, this yields $h_n = n!a_n$.

To see that $h_n = n^n$, we use $g(x) = xe^{g(x)}$, a well-known consequence of the Lagrange Inversion Formula or of combinatorial manipulation of the exponential generating function for rooted trees. Since $f(x) = 1/(1-x)$ as a formal series, $h(x) = 1/(1-g(x))$. On the other hand, $\sum n^n x^n/n! = 1 + xg'(x)$. Thus it suffices to show that $(1+xg'(x))(1-g(x)) = 1$. This follows from computing $g'(x)$, which yields $xg'(x) = g(x) + xg'(x)g(x)$.

Solution II by GCHQ Problems Group, Cheltenham, U.K. We show that the sum equals $n^n/n!$ by counting a set of size n^n in another way, obtaining $n!$ times the desired sum. By Cayley's Formula, the set consisting of rooted trees on the vertex set $\{1, \ldots, n\}$ with one vertex marked has size n^n.

Alternatively, we view each such tree as a list of the subtrees that remain after deleting the edges on the path from the root to the marked vertex. First we partition the n points into subsets, using k_i subsets of size i for $1 \le i \le n$. For each n-tuple $(k_1, \ldots, k_n)$, there are $n!/\left(\prod_i i!^{k_i} k_i!\right)$ ways to do this is. We multiply this by $(\sum k_i)!$ and by $\prod_i (i^{i-1})^{k_i}$ to order the subsets and to place a rooted tree on the elements of each subset. When we assemble the trees, their roots in order form the path from the root to the marked vertex in the full tree. Since we obtain each rooted tree with marked vertex exactly once, we have proved that

$$\sum_{k_1,\ldots,k_n} \frac{n!}{\prod_i i!^{k_i} k_i!} \Big(\sum_i k_i\Big)! \prod_i (i^{i-1})^{k_i} = n^n.$$

Editorial comment. The GCHQ Problems Group noted that omitting the factor $(\sum k_i)!$ yields unordered rooted forests on the n points, which correspond to rooted trees with $n+1$