An Identity Involving Rooted Trees: 10615

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$a^2 = \mu$, and hence $w \succeq (1-\sqrt{\mu})^2/2$. Excluding the wasted sums yields $\mu \preceq 1/2 - w$, and so $2\mu \preceq 1-(1-\sqrt{\mu})^2 = 2\sqrt{\mu} - \mu$. Thus $\sqrt{\mu} \preceq 2/3$ and $\mu \preceq 4/9$.

Editorial comment. John Lindsey proved that $C(m)/m^2 < .499785077 + \epsilon$. Kevin Ford proved that $C(m)/m^2 < .48832 + \epsilon$ and conjectured that the number on the right can be replaced by $\pi/8 < .393$.

Solved also by R. J. Chapman, K. Ford, J. H. Lindsey II, and the proposer.

An Identity Involving Rooted Trees

10615 [1997, 767]. *Proposed by Joaquin Gómez Rey, Alcorcón, Madrid, Spain.* For n a positive integer, evaluate

$$\sum (k_1 + k_2 + \cdots + k_n)! \prod_{i=1}^{n} \frac{i^{(i-1)k_i}}{(k_i!)(i!)^{k_i}}$$

where the summation runs over all n-tuples $(k_1, k_2, \ldots, k_n)$ of nonnegative integers such that $k_1 + 2k_2 + \cdots + nk_n = n$.

Solution I by Anchorage Math Solutions Group, University of Alaska, Anchorage, AK. We show that the given expression a_n equals $n^n/n!$.

Let f, g, h be exponential generating functions with coefficients f_n, g_n, h_n, respectively. When $f(g(x)) = h(x)$, expanding the composition and collecting terms appropriately yields

$$h(x) = \sum_{n=0}^{\infty} n! \left(\sum_{l=1}^{n} f_l \cdot \sum \frac{\prod_{j=1}^{l} g_{\lambda_j}}{\prod_i (k_i!)(i!)^{k_i}} \right) \frac{x^n}{n!},$$

where the innermost sum is over partitions λ of n that have k_i parts of size i for each i and l parts all told. When $f_n = n!$ and $g_n = n^{n-1}$, this yields $h_n = n!a_n$.

To see that $h_n = n^n$, we use $g(x) = xe^{g(x)}$, a well-known consequence of the Lagrange Inversion Formula or of combinatorial manipulation of the exponential generating function for rooted trees. Since $f(x) = 1/(1-x)$ as a formal series, $h(x) = 1/(1-g(x))$. On the other hand, $\sum n^n x^n/n! = 1 + xg'(x)$. Thus it suffices to show that $(1+xg'(x))(1-g(x)) = 1$. This follows from computing $g'(x)$, which yields $xg'(x) = g(x) + xg'(x)g(x)$.

Solution II by GCHQ Problems Group, Cheltenham, U.K. We show that the sum equals $n^n/n!$ by counting a set of size n^n in another way, obtaining $n!$ times the desired sum. By Cayley's Formula, the set consisting of rooted trees on the vertex set $\{1, \ldots, n\}$ with one vertex marked has size n^n.

Alternatively, we view each such tree as a list of the subtrees that remain after deleting the edges on the path from the root to the marked vertex. First we partition the n points into subsets, using k_i subsets of size i for $1 \le i \le n$. For each n-tuple $(k_1, \ldots, k_n)$, there are $n!/\left(\prod_i i!^{k_i} k_i!\right)$ ways to do this is. We multiply this by $(\sum k_i)!$ and by $\prod_i (i^{i-1})^{k_i}$ to order the subsets and to place a rooted tree on the elements of each subset. When we assemble the trees, their roots in order form the path from the root to the marked vertex in the full tree. Since we obtain each rooted tree with marked vertex exactly once, we have proved that

$$\sum_{k_1,\ldots,k_n} \frac{n!}{\prod_i i!^{k_i} k_i!} \left(\sum_i k_i\right)! \prod_i (i^{i-1})^{k_i} = n^n.$$

Editorial comment. The GCHQ Problems Group noted that omitting the factor $(\sum k_i)!$ yields unordered rooted forests on the n points, which correspond to rooted trees with $n+1$

labeled points where point $n+1$ is constrained to be the root. Thus this approach also yields the identity

$$\sum_{k_1,\ldots,k_n} \prod_i \frac{(i^{i-1})^{k_i}}{i!^{k_i} k_i!} = \frac{(n+1)^{n-1}}{n!}.$$

Many solvers reduced the evaluation of the coefficient h_n in Solution 1 to the convolution $\sum_{k=0}^{n-1} \binom{n}{k} k^k (n-k)^{n-k-1}$ and obtained n^n for the sum by manipulating classical identites. The same formula is derived in F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-like Structures*, Cambridge Univ. Press, 1998, as an application of the pointing operation on the species of vertebrates, but it may be proved directly using the objects in Solution 2: Count the rooted trees with marked vertex having $n-k$ vertices in the marked subtree. When $k=0$, the root is the marked vertex, and n^{n-1} counts the rooted trees. When $k>0$, choose a set S of k vertices, a rooted tree with marked vertex on S, and a rooted tree on the remaining vertices. Let the marked vertex of the first tree be the parent of the root x in the second tree and view x as the marked vertex in the full tree.

Solved also by R. Bagby, N. Bansal (India), D. Beckwith, D. Callan, R. J. Chapman (U.K.), W. Chu (France), J. H. Lindsey II, H.-J. Seiffert (Germany), L. Takacs, D. Zeilberger, NSA Problems Group, and the proposer.

Divisors of Sums of Divisors

10617 [1997, 767]. *Proposed by James G. Merickel, Philadelphia, PA.* For a positive integer N, $\sigma(N)$ denotes the sum of the positive divisors of N. Given a positive integer n and a prime p, prove that there exist arbitrarily large sets S of multiples of n with the following property: For some positive integer m, the fraction $\sigma(N)/N$ reduces to a fraction whose denominator is p^m for every $N \in S$.

Solution by John P. Robertson, Berwyn, PA. Factor n as $p^s v$ with v relatively prime to p. Similarly factor $\sigma(v)$ as $p^t w$ with w relatively prime to p. We first consider the case when $p \neq 2$. By Dirichlet's Theorem, there are arbitrarily large sets of primes not congruent to -1 modulo p. Let Q be such a set not containing p or any primes that divide v. Let a be the product of the primes in Q.

Let $u = \phi(av(p-1))$, where ϕ denotes Euler's ϕ-function. Let $r = ku - 1$ with k large enough so that $r > \max\{s, t\}$, and let $m = r - t$. For each positive divisor b of a, the number $N = vp^r b$ is divisible by n because $r > s$. There are $2^{|Q|}$ such divisors b and hence there are $2^{|Q|}$ such N. Thus we need to show only that each fraction $\sigma(N)/N$, reduced, has denominator p^m.

Note that v, p^r, and b are pairwise relatively prime. Hence $\sigma(N)$ is $\sigma(v)\sigma(p^r)\sigma(b)$. Now $\sigma(p^r)$ is $(p^{r+1}-1)/(p-1)$. Because $r+1$ is a multiple of u and p is relatively prime to $av(p-1)$, we find that $(p^{r+1}-1)/(bv(p-1))$ is an integer that is not divisible by p. Also p does not divide the integer $w = \sigma(v)/p^t$, and p does not divide $\sigma(b)$, because no prime in Q is congruent to -1 modulo p. Thus

$$\frac{\sigma(N)}{N} = \frac{\left[\sigma(v)/p^t\right]\left[(p^{r+1}-1)/(bv(p-1))\right]\left[\sigma(b)\right]}{p^m},$$

where each item in brackets is an integer that is not divisible by p.

For the case $p = 2$, we repeat the argument with the following changes: Let Q be any set of odd primes that do not divide v. Use a^2 in place of a and b^2 in place of b, set $u = \phi(a^2 v(p-1))$, and set $N = vp^r b^2$. Since $\sigma(b^2)$ is odd, the argument proceeds in the same way.

Solved also by GCHQ Problems Group and the proposer.