



Divisors of Sums of Divisors: 10617

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labeled points where point $n + 1$ is constrained to be the root. Thus this approach also yields the identity

$$\sum_{k_1, \dots, k_n} \prod_i \frac{(i-1)^{k_i}}{i! k_i!} = \frac{(n+1)^{n-1}}{n!}.$$

Many solvers reduced the evaluation of the coefficient h_n in Solution 1 to the convolution $\sum_{k=0}^{n-1} \binom{n}{k} k^k (n-k)^{n-k-1}$ and obtained n^n for the sum by manipulating classical identities. The same formula is derived in F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-like Structures*, Cambridge Univ. Press, 1998, as an application of the pointing operation on the species of vertebrates, but it may be proved directly using the objects in Solution 2: Count the rooted trees with marked vertex having $n - k$ vertices in the marked subtree. When $k = 0$, the root is the marked vertex, and n^{n-1} counts the rooted trees. When $k > 0$, choose a set S of k vertices, a rooted tree with marked vertex on S , and a rooted tree on the remaining vertices. Let the marked vertex of the first tree be the parent of the root x in the second tree and view x as the marked vertex in the full tree.

Solved also by R. Bagby, N. Bansal (India), D. Beckwith, D. Callan, R. J. Chapman (U.K.), W. Chu (France), J. H. Lindsey II, H.-J. Seiffert (Germany), L. Takacs, D. Zeilberger, NSA Problems Group, and the proposer.

Divisors of Sums of Divisors

10617 [1997, 767]. *Proposed by James G. Merickel, Philadelphia, PA.* For a positive integer N , $\sigma(N)$ denotes the sum of the positive divisors of N . Given a positive integer n and a prime p , prove that there exist arbitrarily large sets S of multiples of n with the following property: For some positive integer m , the fraction $\sigma(N)/N$ reduces to a fraction whose denominator is p^m for every $N \in S$.

Solution by John P. Robertson, Berwyn, PA. Factor n as $p^s v$ with v relatively prime to p . Similarly factor $\sigma(v)$ as $p^t w$ with w relatively prime to p . We first consider the case when $p \neq 2$. By Dirichlet's Theorem, there are arbitrarily large sets of primes not congruent to -1 modulo p . Let Q be such a set not containing p or any primes that divide v . Let a be the product of the primes in Q .

Let $u = \phi(av(p-1))$, where ϕ denotes Euler's ϕ -function. Let $r = ku - 1$ with k large enough so that $r > \max\{s, t\}$, and let $m = r - t$. For each positive divisor b of a , the number $N = vp^r b$ is divisible by n because $r > s$. There are $2^{|Q|}$ such divisors b and hence there are $2^{|Q|}$ such N . Thus we need to show only that each fraction $\sigma(N)/N$, reduced, has denominator p^m .

Note that v , p^r , and b are pairwise relatively prime. Hence $\sigma(N)$ is $\sigma(v)\sigma(p^r)\sigma(b)$. Now $\sigma(p^r)$ is $(p^{r+1} - 1)/(p - 1)$. Because $r + 1$ is a multiple of u and p is relatively prime to $av(p - 1)$, we find that $(p^{r+1} - 1)/(bv(p - 1))$ is an integer that is not divisible by p . Also p does not divide the integer $w = \sigma(v)/p^t$, and p does not divide $\sigma(b)$, because no prime in Q is congruent to -1 modulo p . Thus

$$\frac{\sigma(N)}{N} = \frac{\left[\sigma(v)/p^t\right] \left[(p^{r+1} - 1)/(bv(p - 1))\right] \left[\sigma(b)\right]}{p^m},$$

where each item in brackets is an integer that is not divisible by p .

For the case $p = 2$, we repeat the argument with the following changes: Let Q be any set of odd primes that do not divide v . Use a^2 in place of a and b^2 in place of b , set $u = \phi(a^2 v(p - 1))$, and set $N = vp^r b^2$. Since $\sigma(b^2)$ is odd, the argument proceeds in the same way.

Solved also by GCHQ Problems Group and the proposer.