The Union of Vieta's and Wallis's Products for Pi

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The beautiful infinite product of radicals

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots \tag{1}$$

due to Vieta in 1592 [**2**], is one of the oldest noniterative analytical expressions for π. Wallis's product dating from 1655 [**3**]

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots \tag{2}$$

is also most remarkable. Both are usually included in any list of interesting expressions for π [**1**].

The purpose of this short note is to call attention to the following union of Vieta and Wallis-like products:

$$\frac{2}{\pi} = \prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}}_{(n\ radicals)}$$

$$\times\prod_{n=1}^{\infty}\frac{2^{p+1}n-1}{2^{p+1}n}\cdot\frac{2^{p+1}n+1}{2^{p+1}n}. \tag{3}$$

While (1) and (2) seem unrelated, they are both special cases of a more general double product (3). The first product in (3) consists of the first p factors of Vieta's original infinite product (1). The second product in (3) is a Wallis-like product. We say this because the case $p = 0$ gives us Wallis's original product (2), and for other values of p it is Wallis's product with factors deleted. Notice also that the Wallis-like product in (3) provides us with the error factor needed to make the Vieta product (1) exact when only finitely many factors are used.

Relation (3) yields Vieta's product (1) when p goes to infinity, and Wallis's product (2) when $p = 0$. For each intermediate value of $p = 1, 2, 3, \ldots$ we obtain *united Vieta-Wallis-like products*:

$$p = 0: \quad \frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{9\cdot 11}{10\cdot 10}\,\frac{11\cdot 13}{12\cdot 12}\cdots$$

(Wallis's original product)

$$p = 1: \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{19\cdot 21}{20\cdot 20}\cdots$$

$$p = 2 \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{23\cdot 25}{24\cdot 24}\cdot\frac{31\cdot 33}{32\cdot 32}\cdots$$

$$p = 3: \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{31\cdot 33}{32\cdot 32}\cdot\frac{47\cdot 49}{48\cdot 48}\cdot\frac{63\cdot 65}{64\cdot 64}\cdots$$

$\cdots$

$$p \to \infty: \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

(Vieta's original product)

An examination of these special cases of (3) shows that each time we increase p by one, we insert one new radical factor in the Vieta-like product, and remove alternate factors from the Wallis-like product. The author accidently discovered (3) while trying to derive (1).

To derive (3) we start by applying the double angle formula for the sine function p times to obtain

$$\begin{aligned}\sin\theta &= 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\\ &= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}\\ &= 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}\\ &\ \ \vdots\\ \sin\theta &= 2^p\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\sin\frac{\theta}{2^p}\end{aligned} \tag{4}$$

Next we use the infinite product for the sine function [4], valid for all x,

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = x\prod_{n=1}^{\infty}\left(\frac{\pi n - x}{\pi n}\cdot\frac{xn + x}{\pi n}\right)$$

with $x = \theta/2^p$ to replace the last factor in 4. Dividing by θ gives

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\prod_{n=1}^{\infty}\left(\frac{2^p\pi n - \theta}{2^p\pi n}\cdot\frac{2^p\pi n + \theta}{2^p\pi n}\right). \tag{5}$$

Now express each of the cosine factors in (5) in terms of $\cos\theta$ by repeated use of the half-angle formula for the cosine; here we assume $-\pi/2 \le \theta \le \pi/2$ so that the cosines are never negative.

$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}$$

$$\cos\frac{\theta}{2^2} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}}$$

$$\vdots$$

$$\cos\frac{\theta}{2^p} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}}}} \tag{6}$$

$$(p \textit{ radicals})$$

Combining (6) with (5) we obtain

$$\frac{\sin\theta}{\theta} = \prod_{n=1}^{p} \underbrace{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}}}}}_{(n \textit{ radicals})}$$

$$\times \prod_{n=1}^{\infty} \left(\frac{2^p\pi n - \theta}{2^p\pi n} \cdot \frac{2^p\pi n + \theta}{2^p\pi n} \right) \tag{7}$$

If we set $\theta = \pi/2$ in (7) and simplify we obtain (3). ■

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