10756. *Proposed by Douglas Iannucci, University of the Virgin Islands, St. Thomas, VI.* Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right).$$

10757. *Proposed by Mark Kidwell, United States Naval Academy, Annapolis, MD.* Given integers $a_0, a_1, a_2, \ldots, a_n$ with $a_i \neq 0$ for $i \geq 1$, write $[a_0; a_1, a_2, \ldots, a_n]$ for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}.$$

Every positive rational number has a unique representation as $[a_0; a_1, a_2, \ldots, a_n]$ if we require that $a_0 \geq 0$, $a_i > 0$ for $1 \leq i \leq n-1$, and $a_n > 1$ (we call this the *standard representation*), but it can have other representations $[b_0; b_1, b_2, \ldots, b_m]$ if we permit negative values for some of the b_i or if we permit $b_m = 1$. For example, $11/3 = [3; 1, 2] = [3; 1, 1, 1] = [4; -3]$. Prove or disprove: If r is a positive rational number, $r = [a_0; a_1, a_2, \ldots, a_n]$ is the standard representation, and $r = [b_0; b_1, b_2, \ldots, b_m]$ is another representation, then $a_0 + a_1 + \cdots + a_n \leq |b_0| + |b_1| + \cdots + |b_m|$, with strict inequality if any of the b_i are negative.

10758. *Proposed by Mark Sapir, Vanderbilt University, Nashville, TN.* Prove that the sum of the (decimal) digits of 9^n cannot equal 9 when $n > 2$.

10759. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.* In triangle ABC, let h_a denote the altitude to the side BC and let r_a denote the exradius relative to side BC, i.e., the radius of the circle tangent to the extensions of sides AB and AC and to the side BC externally. Define h_b, h_c, r_b, and r_c correspondingly. Prove that $h_a^n r_a^n + h_b^n r_b^n + h_c^n r_c^n \leq r_a^n r_b^n + r_b^n r_c^n + r_c^n r_a^n$ for any integer n, and determine conditions for equality.

SOLUTIONS

Common Eigenvector of Commuting Matrices

10633 [1997, 975]. *Proposed by Kiran S. Kedlaya, Princeton University, Princeton, NJ.* Let S be a commuting family of n-by-n matrices over an arbitrary field. Suppose the matrices in S have a common eigenvector v, so that $Mv = \lambda_M v$ for all $M \in S$. Prove that the transposes of these matrices also have a common eigenvector with these eigenvalues, that is, a vector w satisfying $M^T w = \lambda_M w$ for all $M \in S$.

Solution by Alain Tissier, Montmermeil, France. Let K be the field. Set $\phi(M) = M - \lambda_M I$ and $\phi(S) = \{\phi(M) \colon M \in S\}$. Thus $\phi(S)$ is a commuting family of $n \times n$ matrices over K having a common nonzero vector v such that $\phi(M)v = 0$ for all $\phi(M) \in \phi(S)$. Since $\phi(M)^T = M^T - \lambda_M I$, we have to prove only that the transposes of the matrices in $\phi(S)$ have a common nonzero vector w satisfying $\phi(M)^T w = 0$ for $\phi(M) \in \phi(S)$. Thus we may suppose that $\lambda_M = 0$ for every M.

If all matrices in S are nilpotent, then the collection of transposes is also a commuting family of nilpotent matrices. In this case there is a nonzero vector w such that $M^T w = 0$ for all $M \in S$ (section 3.3 of J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag, 1972). So we may assume that not all elements of S are nilpotent.

We proceed by induction on n. When $n = 1$ all the matrices are zero, so the conclusion is true. Take $n > 1$, and suppose the result is true for h-by-h matrices for each $h < n$. Let N

be a nonnilpotent element of S. Let W be the set of all vectors x such that $N^k x = 0$ for some $k \geq 0$. By finite-dimensionality, there is a fixed k such that $N^k x = 0$ for all $x \in W$. So $v \in W$, W is a subspace, and $K^n = W \oplus U$, where U is the range of the mapping $x \mapsto N^k x$. Now if $M \in S$, then M commutes with N, and the descriptions of W and U show that they are invariant under M. Let m be the dimension of W, let $\mathcal{B}'$ be a basis of W, and let $\mathcal{B}''$ be a basis of U. For each $M \in S$, let M' be the $\mathcal{B}'$-representation of M restricted to W and let M'' be the $\mathcal{B}''$-representation of M restricted to U. Then there exists a nonsingular $n \times n$ matrix P such that $P^{-1}MP = \begin{bmatrix} M' & 0 \\ 0 & M'' \end{bmatrix}$ for all $M \in S$. Let S' be the set of the matrices M'. Then S' is a family of $m \times m$ commuting matrices having a common nonzero vector v' such that $M'v' = 0$ for each $M' \in S'$. By the induction hypothesis there exists a nonzero vector w' such that $M'^T w' = 0$ for each $M' \in S'$. The vector $(P^T)^{-1} \begin{bmatrix} w' \\ 0 \end{bmatrix}$ solves the problem.

Solved also by R. J. Chapman (U. K.), D. Huang, J. H. Lindsey II, G. Sansigre Vidal (Spain), GCHQ Problems Group (U. K.), and the proposer.

Reflected Concurrent Lines

10637 [1998, 68]. *Proposed by C. F. Parry, Exmouth, Devon, United Kingdom.* Suppose triangle ABC has circumcircle Γ, circumcenter O, and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reflections of α, β, γ in the sides BC, CA, AB, respectively.
(a) Show that α', β', γ' are concurrent if and only if α, β, γ are parallel to the Euler line OH.
(b) Suppose that α', β', γ' are concurrent at the point P. Show that Γ bisects OP.

Solution by Robert L. Young, Osterville, MA. Take Γ to be the unit circle $z\bar{z} = 1$ in the complex plane and rotate ABC about O so that $\arg H = 0$. Assume $H \neq 0$ for now, so the Euler line exists and is the real axis. Choose $\theta_3 > \theta_2 > \theta_1 > 0$ so that $A = e^{i\theta_1}$, $B = e^{i\theta_2}$, and $C = e^{i\theta_3}$, and let $M = e^{i\theta}$, where $\theta \in [0, \pi)$ is the angle of inclination of the lines α, β, γ.
(a) The reflection z' of a complex number z through the line containing B and C is determined as follows. Apply the linear transformation $\tau(z) = (z - B)\overline{(C - B)}$, which takes B and C and therefore the line BC to the real axis. Since reflection in the real axis is conjugation,

$$z' = \tau^{-1}(\overline{\tau(z)}) = \frac{(\overline{z - B})(C - B)}{(\overline{C} - \overline{B})} \frac{BC}{BC} + B = -BC\,\bar{z} + B + C,$$

and the reflection of A through line BC is

$$A' = -BC\,\overline{A} + B + C. \tag{1}$$

Any $z \neq A'$ on α' satisfies the equation

$$\frac{z - A'}{\bar{z} - \overline{A'}} = e^{2i \arg \alpha'}. \tag{2}$$

Since the perpendicular bisector of line BC passes through O and $\exp(i(\theta_2 + \theta_3)/2)$, we have $\arg(C - B) \equiv (\theta_2 + \theta_3)/2 - \pi/2$ modulo π. By the definition of α', $\arg\alpha' + \arg\alpha \equiv 2\arg(C - B) \equiv \theta_2 + \theta_3 - \pi$ modulo 2π, so $e^{2i \arg \alpha'} = e^{i(2\theta_2 + 2\theta_3 - 2\theta)} = B^2C^2\overline{M}^2$. Substituting (1) into (2), we conclude that α' has equation

$$z = \overline{M}^2 C^2 B^2 \left(\bar{z} + A\overline{B}\,\overline{C} - \overline{B} - \overline{C}\right) - BC\overline{A} + B + C.$$