be a nonnilpotent element of S. Let W be the set of all vectors x such that $N^k x = 0$ for some $k \geq 0$. By finite-dimensionality, there is a fixed k such that $N^k x = 0$ for all $x \in W$. So $v \in W$, W is a subspace, and $K^n = W \oplus U$, where U is the range of the mapping $x \mapsto N^k x$. Now if $M \in S$, then M commutes with N, and the descriptions of W and U show that they are invariant under M. Let m be the dimension of W, let $\mathcal{B}'$ be a basis of W, and let $\mathcal{B}''$ be a basis of U. For each $M \in S$, let M' be the $\mathcal{B}'$-representation of M restricted to W and let M'' be the $\mathcal{B}''$-representation of M restricted to U. Then there exists a nonsingular $n \times n$ matrix P such that $P^{-1}MP = \begin{bmatrix} M' & 0 \\ 0 & M'' \end{bmatrix}$ for all $M \in S$. Let S' be the set of the matrices M'. Then S' is a family of $m \times m$ commuting matrices having a common nonzero vector v' such that $M'v' = 0$ for each $M' \in S'$. By the induction hypothesis there exists a nonzero vector w' such that $M'^T w' = 0$ for each $M' \in S'$. The vector $(P^T)^{-1} \begin{bmatrix} w' \\ 0 \end{bmatrix}$ solves the problem.

Solved also by R. J. Chapman (U. K.), D. Huang, J. H. Lindsey II, G. Sansigre Vidal (Spain), GCHQ Problems Group (U. K.), and the proposer.

Reflected Concurrent Lines

10637 [1998, 68]. *Proposed by C. F. Parry, Exmouth, Devon, United Kingdom.* Suppose triangle ABC has circumcircle Γ, circumcenter O, and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reflections of α, β, γ in the sides BC, CA, AB, respectively.

(**a**) Show that α', β', γ' are concurrent if and only if α, β, γ are parallel to the Euler line OH.

(**b**) Suppose that α', β', γ' are concurrent at the point P. Show that Γ bisects OP.

Solution by Robert L. Young, Osterville, MA. Take Γ to be the unit circle $z\overline{z} = 1$ in the complex plane and rotate ABC about O so that $\arg H = 0$. Assume $H \neq 0$ for now, so the Euler line exists and is the real axis. Choose $\theta_3 > \theta_2 > \theta_1 > 0$ so that $A = e^{i\theta_1}$, $B = e^{i\theta_2}$, and $C = e^{i\theta_3}$, and let $M = e^{i\theta}$, where $\theta \in [0, \pi)$ is the angle of inclination of the lines α, β, γ.

(**a**) The reflection z' of a complex number z through the line containing B and C is determined as follows. Apply the linear transformation $\tau(z) = (z - B)\overline{(C - B)}$, which takes B and C and therefore the line BC to the real axis. Since reflection in the real axis is conjugation,

$$z' = \tau^{-1}(\overline{\tau(z)}) = \frac{(\overline{z - B})(C - B)}{(\overline{C} - \overline{B})} \frac{BC}{BC} + B = -BC\,\overline{z} + B + C,$$

and the reflection of A through line BC is

$$A' = -BC\,\overline{A} + B + C. \tag{1}$$

Any $z \neq A'$ on α' satisfies the equation

$$\frac{z - A'}{\overline{z} - \overline{A'}} = e^{2i \arg \alpha'}. \tag{2}$$

Since the perpendicular bisector of line BC passes through O and $\exp(i(\theta_2 + \theta_3)/2)$, we have $\arg(C - B) \equiv (\theta_2 + \theta_3)/2 - \pi/2$ modulo π. By the definition of α', $\arg \alpha' + \arg \alpha \equiv 2\arg(C - B) \equiv \theta_2 + \theta_3 - \pi$ modulo 2π, so $e^{2i \arg \alpha'} = e^{i(2\theta_2 + 2\theta_3 - 2\theta)} = B^2 C^2 \overline{M}^2$. Substituting (1) into (2), we conclude that α' has equation

$$z = \overline{M}^2 C^2 B^2 \left(\overline{z} + A\overline{B}\,\overline{C} - \overline{B} - \overline{C}\right) - BC\overline{A} + B + C.$$

It is convenient to note that $A+B+C=H$ and is therefore real and to write $K=ABC$, so that $AB+BC+CA=K\overline{H}=KH$. With this notation, the equation becomes $z=\overline{M}^2K^2\overline{A}^2(\overline{z}+(A-C-B)\overline{B}\,\overline{C})+(AB+AC-BC)\overline{A}$, or

$$z=K\big(\overline{M}^2K\overline{z}-2\big)\overline{A}^2-\big(\overline{M}^2-1\big)KH\overline{A}+2\overline{M}^2K.$$

Similarly, the equation of β' is

$$z=K\big(\overline{M}^2K\overline{z}-2\big)\overline{B}^2-\big(\overline{M}^2-1\big)KH\,\overline{B}+2\overline{M}^2K.$$

Let z_C denote point of intersection, if any, of α' and β' and similarly for z_A and z_B. Solving for z_C from these two equations, we get $K\big(\overline{M}^2K\overline{z_C}-2\big)\overline{A}^2-\big(\overline{M}^2-1\big)KH\,\overline{A}=K\big(\overline{M}^2K\overline{z_C}-2\big)\overline{B}^2-\big(\overline{M}^2-1\big)KH\,\overline{B}$, so $K\big(\overline{A}^2-\overline{B}^2\big)\big(\overline{M}^2K\overline{z_C}-2\big)=\big(\overline{A}-\overline{B}\big)\big(\overline{M}^2-1\big)KH$, and

$$\big(\overline{M}^2K\overline{z_C}-2\big)\big(\overline{A}+\overline{B}\big)=(\overline{M}^2-1)H.$$

Similarly,

$$\big(\overline{M}^2K\overline{z_B}-2\big)\big(\overline{A}+\overline{C}\big)=\big(\overline{M}^2K\overline{z_A}-2\big)\big(\overline{B}+\overline{C}\big)=\big(\overline{M}^2-1\big)H.$$

Suppose α', β', γ' are concurrent at P. Then $\big(\overline{A}+\overline{B}\big)\big(\overline{M}^2K\overline{P}-2\big)$, $\big(\overline{B}+\overline{C}\big)\big(\overline{M}^2K\overline{P}-2\big)$, and $\big(\overline{C}+\overline{A}\big)\big(\overline{M}^2K\overline{P}-2\big)$ all equal $\big(\overline{M}^2-1\big)H$. Multiply the first of these equations by $\overline{B}+\overline{C}$, multiply the second by $\overline{A}+\overline{B}$, and then subtract to obtain $0=\big(\overline{M}^2-1\big)H\big(\overline{A}-\overline{C}\big)$. Since $A\neq C$ and $H\neq 0$, we have $M^2=1$ and $\theta=0$. So α, β, γ are parallel to the Euler line as claimed. Conversely, if α, β, γ are parallel to the Euler line, then $M^2=1$, and $z_A=z_B=z_C=P=2K$ satisfy the equations for α', β', γ', so these are concurrent.

If $H=0$, there is no Euler line. In this case, α', β', and γ' concur at $P=2K\,\overline{M}^2$.
(b) Since $P=2K=2ABC$, we have $|P|=2$. Therefore $|(O+P)/2|=1$ and $(O+P)/2$ is on Γ.

Solved also by J. Anglesio (France), M. Benedicty, N. Lakshmanan, and V. Schindler (Germany).

A Constrained Maximization

10646 [1998, 176]. *Proposed by Hassan Ali Shah Ali, Teheran, Iran.* Find the maximum of $\prod_{i=1}^{n}(1-x_i)$ over all nonnegative $x_1, x_2, \ldots, x_n$ with $\sum_{i=1}^{n}x_i^2=1$.

Solution by Patrick A. Staley, Southwestern College, Chula Vista, CA. When $n=1$, the constraint requires $x_1=1$, and the maximum value is 0. So assume $n\geq 2$. We show that the maximum is $3/2-\sqrt{2}\approx 0.0858$, and it occurs when two of the x_i's are $1/\sqrt{2}$ and the others are 0.

Let $x_1, x_2, \ldots, x_n$ be an optimal solution. If x and y are any two of the x_i's, then they satisfy a two-element subproblem: maximize $(1-x)(1-y)$ under the constraints $x\geq 0$, $y\geq 0$, and $x^2+y^2=k^2$ for a given positive $k\leq 1$. To solve this, note that $dy/dx=-x/y$, so

$$\frac{d\big((1-x)(1-y)\big)}{dx}=-(1-y)-(1-x)\frac{dy}{dx}=\frac{(x-y)(1-x-y)}{y}.$$

If this vanishes, then $(x+y-1)(x-y)=0$. There are three possibilities for the global maximum of $(1-x)(1-y)$:
(1) endpoints, $x=0$, $y=k$ (or vice versa), so $(1-x)(1-y)=(1-k)$;
(2) $y=x$, so $x=y=k/\sqrt{2}$, $(1-x)(1-y)=(1-k/\sqrt{2})^2$; or
(3) $y=1-x$, so $x, y=(1\pm\sqrt{2k^2-1})/2$ and $(1-x)(1-y)=(1-k^2)/2$.
Case **(3)** may be discarded, since $(1-k^2)/2\leq(1-k)$ for all k. If $k<2(\sqrt{2}-1)\approx 0.828$ then case **(1)** is maximal; otherwise, case **(2)** is maximal.