A Constrained Maximization: 10646

Hassan Ali Shah Ali; Patrick A. Staley

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It is convenient to note that $A+B+C=H$ and is therefore real and to write $K=ABC$, so that $AB+BC+CA = K\overline{H} = KH$. With this notation, the equation becomes $z=\overline{M}^2K^2\overline{A}^2(\overline{z}+(A-C-B)\overline{B}\,\overline{C})+(AB+AC-BC)\overline{A}$, or

$$z=K\big(\overline{M}^2K\overline{z}-2\big)\overline{A}^2-\big(\overline{M}^2-1\big)KH\overline{A}+2\overline{M}^2K.$$

Similarly, the equation of β' is

$$z=K\big(\overline{M}^2K\overline{z}-2\big)\overline{B}^2-\big(\overline{M}^2-1\big)KH\,\overline{B}+2\overline{M}^2K.$$

Let z_C denote point of intersection, if any, of α' and β' and similarly for z_A and z_B. Solving for z_C from these two equations, we get $K\big(\overline{M}^2K\overline{z_C}-2\big)\overline{A}^2-\big(\overline{M}^2-1\big)KH\,\overline{A}=K\big(\overline{M}^2K\overline{z_C}-2\big)\overline{B}^2-\big(\overline{M}^2-1\big)KH\,\overline{B}$, so $K\big(\overline{A}^2-\overline{B}^2\big)\big(\overline{M}^2K\overline{z_C}-2\big)=\big(\overline{A}-\overline{B}\big)\big(\overline{M}^2-1\big)KH$, and

$$\big(\overline{M}^2K\overline{z_C}-2\big)\big(\overline{A}+\overline{B}\big)=(\overline{M}^2-1)H.$$

Similarly,

$$\big(\overline{M}^2K\overline{z_B}-2\big)\big(\overline{A}+\overline{C}\big)=\big(\overline{M}^2K\overline{z_A}-2\big)\big(\overline{B}+\overline{C}\big)=\big(\overline{M}^2-1\big)H.$$

Suppose α', β', γ' are concurrent at P. Then $\big(\overline{A}+\overline{B}\big)\big(\overline{M}^2K\overline{P}-2\big)$, $\big(\overline{B}+\overline{C}\big)\big(\overline{M}^2K\overline{P}-2\big)$, and $\big(\overline{C}+\overline{A}\big)\big(\overline{M}^2K\overline{P}-2\big)$ all equal $\big(\overline{M}^2-1\big)H$. Multiply the first of these equations by $\overline{B}+\overline{C}$, multiply the second by $\overline{A}+\overline{B}$, and then subtract to obtain $0=\big(\overline{M}^2-1\big)H\big(\overline{A}-\overline{C}\big)$. Since $A\neq C$ and $H\neq 0$, we have $M^2=1$ and $\theta=0$. So α, β, γ are parallel to the Euler line as claimed. Conversely, if α, β, γ are parallel to the Euler line, then $M^2=1$, and $z_A=z_B=z_C=P=2K$ satisfy the equations for α', β', γ', so these are concurrent.

If $H=0$, there is no Euler line. In this case, α', β', and γ' concur at $P=2K\,\overline{M}^2$.

(b) Since $P=2K=2ABC$, we have $|P|=2$. Therefore $|(O+P)/2|=1$ and $(O+P)/2$ is on Γ.

Solved also by J. Anglesio (France), M. Benedicty, N. Lakshmanan, and V. Schindler (Germany).

A Constrained Maximization

10646 [1998, 176]. *Proposed by Hassan Ali Shah Ali, Teheran, Iran.* Find the maximum of $\prod_{i=1}^n(1-x_i)$ over all nonnegative $x_1, x_2, \dots, x_n$ with $\sum_{i=1}^n x_i^2=1$.

Solution by Patrick A. Staley, Southwestern College, Chula Vista, CA. When $n=1$, the constraint requires $x_1=1$, and the maximum value is 0. So assume $n\geq 2$. We show that the maximum is $3/2-\sqrt{2}\approx 0.0858$, and it occurs when two of the x_i's are $1/\sqrt{2}$ and the others are 0.

Let $x_1, x_2, \dots, x_n$ be an optimal solution. If x and y are any two of the x_i's, then they satisfy a two-element subproblem: maximize $(1-x)(1-y)$ under the constraints $x\geq 0$, $y\geq 0$, and $x^2+y^2=k^2$ for a given positive $k\leq 1$. To solve this, note that $dy/dx=-x/y$, so

$$\frac{d\big((1-x)(1-y)\big)}{dx}=-(1-y)-(1-x)\frac{dy}{dx}=\frac{(x-y)(1-x-y)}{y}.$$

If this vanishes, then $(x+y-1)(x-y)=0$. There are three possibilities for the global maximum of $(1-x)(1-y)$:

(1) endpoints, $x=0$, $y=k$ (or vice versa), so $(1-x)(1-y)=(1-k)$;

(2) $y=x$, so $x=y=k/\sqrt{2}$, $(1-x)(1-y)=(1-k/\sqrt{2})^2$; or

(3) $y=1-x$, so $x, y=(1\pm\sqrt{2k^2-1})/2$ and $(1-x)(1-y)=(1-k^2)/2$.

Case **(3)** may be discarded, since $(1-k^2)/2\leq(1-k)$ for all k. If $k<2(\sqrt{2}-1)\approx 0.828$ then case **(1)** is maximal; otherwise, case **(2)** is maximal.

Now consider a three-element subproblem. Let x, y, z be any three of the x_i's. They maximize $(1-x)(1-y)(1-z)$ subject to $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2+y^2+z^2=h^2$ for a given positive $h \leq 1$. Now the largest element must be at least $h/\sqrt{3}$, so the other two elements solve the two-element subproblem with $k \leq \sqrt{2/3}\,h < 0.828$, so for that subproblem case **(1)** is maximal, and thus one of the variables must be 0.

Since one of every three variables must be 0, there can be at most two nonzero variables. Those two solve the two-element problem with $k = 1$, so the maximum occurs in case **(2)** and the maximum is $(1-1/\sqrt{2})^2 = 3/2-\sqrt{2}$.

Editorial comment. There were a large number of incorrect solutions. Many of these used Lagrange multipliers to find a local maximum for the function in question, but ignored the possibility of a global maximum occurring at a boundary point, as it does when $n \geq 3$.

Solved also by R. A. Agnew, Z. Ahmed & A. N. Joseph & M. A. Prasad (India), R. Barbara, M. Benedicty, B. Borchers, P. Budney, R. J. Chapman (U. K.), C. Georghiou (Greece), G. Keselman, A. Kundgen, J. H. Lindsey II, S. Pedersen (Denmark), C. Popescu (Belgium), A. Rosenthal, W. J. Seaman, H. A. Steinberg, A. Stenger, J. Vandergriff, J. T. Ward, Q. Yao, GCHQ Problems Group, IUTS Problems Group, NSA Problems Group, and the proposer.

A Pólya-Szegö Exercise Revisited

10650 [1998, 271]. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* For $n \geq 2$, let

$$a_n = \frac{(n^2+1)(n^2+2)\cdots(n^2+n)}{(n^2-1)(n^2-2)\cdots(n^2-n)}.$$

Then $\lim_{n\to\infty} a_n = e$, by exercise 55 in G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, 1972. Show that $\lim_{n\to\infty} n(a_n - e) = e$.

Solution by William F. Trench, Trinity University, San Antonio, TX. By Taylor's Theorem applied to $f(x) = \log\big((1+x)/(1-x)\big)$,

$$|f(x) - 2x| \leq \frac{2x^3}{(1-x^2)^2} \qquad \text{for } 0 < x < 1.$$

Since $\log a_n = \sum_{j=1}^n f\left(j/n^2\right)$, we have

$$\left|\log a_n - 1 - \frac{1}{n}\right| \leq \sum_{j=1}^{n} \left| f\left(\frac{j}{n^2}\right) - \frac{2j}{n^2}\right| \leq \frac{2}{n^6(1-1/n)^2}\sum_{j=1}^{n} j^3 = O\left(\frac{1}{n^2}\right)$$

as $n \to \infty$. Therefore

$$a_n = e\exp\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = e\left(1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right),$$

which implies that $n(a_n - e) = e + O(1/n)$.

Editorial comment. Several solvers obtained additional terms in the asymptotic expansion of $\log a_n$ and thus of a_n. Douglas B. Tyler computed the former completely in terms of the Bernoulli numbers $B_0, B_1, B_2, \ldots$ as follows:

$$\log a_n = 1 + \frac{1}{2}\log\frac{n+1}{n-1} + \sum_{i=1}^{N-1}\frac{1}{n^{2i}}\left(\sum_{j=\lfloor\frac{i+1}{2}\rfloor}^{i} \frac{B_{2i-2j}}{(2j+1)(j+1)}\binom{2j+2}{2i-2j}\right) + O\left(\frac{1}{n^{2N}}\right).$$

In a different direction, William A. Newcomb proved that the original conclusion holds for

$$a_n = \prod_{k=1}^{n} \frac{n + f(k/n)}{n - f(k/n)},$$