Now consider a three-element subproblem. Let x, y, z be any three of the x_i's. They maximize $(1-x)(1-y)(1-z)$ subject to $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2+y^2+z^2=h^2$ for a given positive $h \leq 1$. Now the largest element must be at least $h/\sqrt{3}$, so the other two elements solve the two-element subproblem with $k \leq \sqrt{2/3}\,h < 0.828$, so for that subproblem case **(1)** is maximal, and thus one of the variables must be 0.

Since one of every three variables must be 0, there can be at most two nonzero variables. Those two solve the two-element problem with $k = 1$, so the maximum occurs in case **(2)** and the maximum is $(1-1/\sqrt{2})^2 = 3/2-\sqrt{2}$.

Editorial comment. There were a large number of incorrect solutions. Many of these used Lagrange multipliers to find a local maximum for the function in question, but ignored the possibility of a global maximum occurring at a boundary point, as it does when $n \geq 3$.

Solved also by R. A. Agnew, Z. Ahmed & A. N. Joseph & M. A. Prasad (India), R. Barbara, M. Benedicty, B. Borchers, P. Budney, R. J. Chapman (U. K.), C. Georghiou (Greece), G. Keselman, A. Kundgen, J. H. Lindsey II, S. Pedersen (Denmark), C. Popescu (Belgium), A. Rosenthal, W. J. Seaman, H. A. Steinberg, A. Stenger, J. Vandergriff, J. T. Ward, Q. Yao, GCHQ Problems Group, IUTS Problems Group, NSA Problems Group, and the proposer.

A Pólya-Szegö Exercise Revisited

10650 [1998, 271]. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* For $n \geq 2$, let

$$a_n = \frac{(n^2+1)(n^2+2)\cdots(n^2+n)}{(n^2-1)(n^2-2)\cdots(n^2-n)}.$$

Then $\lim_{n\to\infty} a_n = e$, by exercise 55 in G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, 1972. Show that $\lim_{n\to\infty} n(a_n - e) = e$.

Solution by William F. Trench, Trinity University, San Antonio, TX. By Taylor's Theorem applied to $f(x) = \log\big((1+x)/(1-x)\big)$,

$$|f(x) - 2x| \leq \frac{2x^3}{(1-x^2)^2} \qquad \text{for } 0 < x < 1.$$

Since $\log a_n = \sum_{j=1}^n f\left(j/n^2\right)$, we have

$$\left|\log a_n - 1 - \frac{1}{n}\right| \leq \sum_{j=1}^n \left| f\left(\frac{j}{n^2}\right) - \frac{2j}{n^2}\right| \leq \frac{2}{n^6(1-1/n)^2}\sum_{j=1}^n j^3 = O\left(\frac{1}{n^2}\right)$$

as $n \to \infty$. Therefore

$$a_n = e\exp\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = e\left(1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right),$$

which implies that $n(a_n - e) = e + O(1/n)$.

Editorial comment. Several solvers obtained additional terms in the asymptotic expansion of $\log a_n$ and thus of a_n. Douglas B. Tyler computed the former completely in terms of the Bernoulli numbers $B_0, B_1, B_2, \ldots$ as follows:

$$\log a_n = 1 + \frac{1}{2}\log\frac{n+1}{n-1} + \sum_{i=1}^{N-1}\frac{1}{n^{2i}}\left(\sum_{j=\lfloor\frac{i+1}{2}\rfloor}^{i}\frac{B_{2i-2j}}{(2j+1)(j+1)}\binom{2j+2}{2i-2j}\right) + O\left(\frac{1}{n^{2N}}\right).$$

In a different direction, William A. Newcomb proved that the original conclusion holds for

$$a_n = \prod_{k=1}^n \frac{n + f(k/n)}{n - f(k/n)},$$

where f is an arbitrary C^2 function on $[0, 1]$.

Solved also by Z. Ahmed & M.A. Prasad (India), J. Anglesio (France), G. L. Body (U. K.), P. Bracken (Canada), R. J. Chapman (U. K.), R. Cuculiere (France), J. Deutsch, K. P. Hart (The Netherlands), G. Keselman, J. H. Lindsey II, V. Lucic (Canada), W. A. Newcomb, M. Omarjee (France), K. Schilling, H.-J. Seiffert (Germany), P. Simeonov, N. C. Singer, I. Sofair, A. Stadler (Switzerland), A. Stenger, D. B. Tyler, J. H. van Lint (The Netherlands), J. Wimp, GCHQ Problems Group (U. K.), and the proposer.

Harmonic Products of Harmonic Functions

10651 [1998, 271]. *Proposed by W. K. Hayman, Imperial College, London, U. K.* If u_1 and u_2 are nonconstant real functions of two variables, and if u_1, u_2, and $u_1 u_2$ are all harmonic in a simply connected plane domain D, prove that $u_2 = av_1 + b$, where v_1 is a harmonic conjugate of u_1 in D, and a and b are real constants.

Solution by Tewodros Amdeberhan, DeVry Institute, North Brunswick, NJ. In $\mathbb{R}^2$, we write w_x and w_y for $\partial w/\partial x$ and $\partial w/\partial y$. Let $f = u_1 + iv_1$. Since f is analytic, f^2 is analytic, and hence $2u_1v_1 = \mathrm{Im}(f^2)$ is harmonic. Since

$$\triangle(u_1u_2) = \triangle u_1 + \triangle u_2 + 2\nabla u_1 \cdot \nabla u_2 \text{ and } \triangle(u_1v_1) = \triangle u_1 + \triangle v_1 + 2\nabla u_1 \cdot \nabla v_1,$$

it follows from the hypotheses that both vectors ∇u_2 and ∇v_1 are orthogonal to ∇u_1 in R^2. Thus

$$\nabla u_2 = a\nabla v_1, \qquad (1)$$

for some real function $a = a(x, y)$. Consequently, $\triangle u_2 = a \triangle v_1 + \nabla a \cdot \nabla v_1$, and so

$$\nabla v_1 \cdot (a_x, a_y) = 0. \qquad (2)$$

Rewriting (1) in terms of components yields $(u_2)_x = a(v_1)_x$ and $(u_2)_y = a(v_1)_y$. Differentiating with respect to y and x, respectively, we get

$$(u_2)_{xy} = a_y(v_1)_x + a(v_1)_{xy} \text{ and } (u_2)_{yx} = a_x(v_1)_y + a(v_1)_{yx}.$$

This shows that

$$\nabla v_1 \cdot (a_y, -a_x) = 0. \qquad (3)$$

Combining (2) and (3) gives $\nabla a \equiv 0$, so a is a constant function. This in turn implies that $\nabla(u_2 - av_1) = \nabla u_2 - a\nabla v_1 \equiv 0$, proving that $u_2 - av_1$ is a constant.

Editorial comment. Irl C. Bivens notes that the " $+\,b$" may be eliminated in the statement of the problem if we are allowed to choose which harmonic conjugate v_1 of u_1 is to be used. He also notes that "simply connected" is not needed in the statement, since the other conditions of the problem imply the existence of a harmonic conjugate.

Solved also by K. F. Andersen (Canada), J. Anglesio (France), I. C. Bivens, R. J. Chapmen (U. K.), R. Govindaraj (India), M. Gruber, R. Mortini (France), I. Netuka (Czech Republic), D. E. Tepper & J. Huntley, W. F. Trench, E. I. Verriest, and the proposer.

Large Values of Tangent

10656 [1998, 366]. *Proposed by David P. Bellamy and Felix Lazebnik, University of Delaware, Newark, DE, and Jeffrey Lagarias, AT&T Laboratories, Florham Park, NJ.*
(a) Show that there are infinitely many positive integers n such that $|\tan n| > n$.
(b) Show that there are infinitely many positive integers n such that $\tan n > n/4$.

Solution by Stephen M. Gagola, Jr., Kent State University, Kent, OH. We use the notation $\alpha = [a_0; a_1, a_2, \ldots]$ to represent the continued fraction expansion of the irrational number

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} .$$