Large Values of Tangent: 10656

David P. Bellamy; Felix Lazebnik; Jeffrey Lagarias; Stephen M. Gagola, Jr.

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where f is an arbitrary C^2 function on $[0, 1]$.

Solved also by Z. Ahmed & M.A. Prasad (India), J. Anglesio (France), G. L. Body (U. K.), P. Bracken (Canada), R. J. Chapman (U. K.), R. Cuculiere (France), J. Deutsch, K. P. Hart (The Netherlands), G. Keselman, J. H. Lindsey II, V. Lucic (Canada), W. A. Newcomb, M. Omarjee (France), K. Schilling, H.-J. Seiffert (Germany), P. Simeonov, N. C. Singer, I. Sofair, A. Stadler (Switzerland), A. Stenger, D. B. Tyler, J. H. van Lint (The Netherlands), J. Wimp, GCHQ Problems Group (U. K.), and the proposer.

Harmonic Products of Harmonic Functions

10651 [1998, 271]. *Proposed by W. K. Hayman, Imperial College, London, U. K.* If u_1 and u_2 are nonconstant real functions of two variables, and if u_1, u_2, and $u_1 u_2$ are all harmonic in a simply connected plane domain D, prove that $u_2 = av_1 + b$, where v_1 is a harmonic conjugate of u_1 in D, and a and b are real constants.

Solution by Tewodros Amdeberhan, DeVry Institute, North Brunswick, NJ. In $\mathbb{R}^2$, we write w_x and w_y for $\partial w/\partial x$ and $\partial w/\partial y$. Let $f = u_1 + iv_1$. Since f is analytic, f^2 is analytic, and hence $2u_1v_1 = \operatorname{Im}(f^2)$ is harmonic. Since

$$\triangle(u_1u_2) = \triangle u_1 + \triangle u_2 + 2\nabla u_1 \cdot \nabla u_2 \text{ and } \triangle\,(u_1v_1) = \triangle u_1 + \triangle v_1 + 2\nabla u_1 \cdot \nabla v_1,$$

it follows from the hypotheses that both vectors ∇u_2 and ∇v_1 are orthogonal to ∇u_1 in R^2. Thus

$$\nabla u_2 = a\nabla v_1, \tag{1}$$

for some real function $a = a(x, y)$. Consequently, $\triangle u_2 = a \triangle v_1 + \nabla a \cdot \nabla v_1$, and so

$$\nabla v_1 \cdot (a_x, a_y) = 0. \tag{2}$$

Rewriting (1) in terms of components yields $(u_2)_x = a(v_1)_x$ and $(u_2)_y = a(v_1)_y$. Differentiating with respect to y and x, respectively, we get

$$(u_2)_{xy} = a_y(v_1)_x + a(v_1)_{xy} \text{ and } (u_2)_{yx} = a_x(v_1)_y + a(v_1)_{yx}.$$

This shows that

$$\nabla v_1 \cdot (a_y, -a_x) = 0. \tag{3}$$

Combining (2) and (3) gives $\nabla a \equiv 0$, so a is a constant function. This in turn implies that $\nabla(u_2 - av_1) = \nabla u_2 - a\nabla v_1 \equiv 0$, proving that $u_2 - av_1$ is a constant.

Editorial comment. Irl C. Bivens notes that the " $+\, b$" may be eliminated in the statement of the problem if we are allowed to choose which harmonic conjugate v_1 of u_1 is to be used. He also notes that "simply connected" is not needed in the statement, since the other conditions of the problem imply the existence of a harmonic conjugate.

Solved also by K. F. Andersen (Canada), J. Anglesio (France), I. C. Bivens, R. J. Chapmen (U. K.), R. Govindaraj (India), M. Gruber, R. Mortini (France), I. Netuka (Czech Republic), D. E. Tepper & J. Huntley, W. F. Trench, E. I. Verriest, and the proposer.

Large Values of Tangent

10656 [1998, 366]. *Proposed by David P. Bellamy and Felix Lazebnik, University of Delaware, Newark, DE, and Jeffrey Lagarias, AT&T Laboratories, Florham Park, NJ.*
(a) Show that there are infinitely many positive integers n such that $|\tan n| > n$.
(b) Show that there are infinitely many positive integers n such that $\tan n > n/4$.

Solution by Stephen M. Gagola, Jr., Kent State University, Kent, OH. We use the notation $\alpha = [a_0; a_1, a_2, \ldots]$ to represent the continued fraction expansion of the irrational number

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}\ .$$

The convergents $h_i/k_i = [a_0; a_1, a_2, \ldots, a_i]$ have the property that the sequences $\{h_i\}$ and $\{k_i\}$ satisfy the same recurrence relation $x_i = a_i x_{i-1} + x_{i-2}$ (but with different initial conditions). From the theory of continued fractions (see for example Kumandure and Romero, *Number Theory with Computer Applications*, Prentice Hall, 1998; especially Chapters 11 and 13), we have $h_{i-1}k_i - h_i k_{i-1} = (-1)^i$, h_{2i}/k_{2i} increases to α, and h_{2i+1}/k_{2i+1} decreases to α. In particular, the interval whose endpoints are h_i/k_i and h_{i+1}/k_{i+1} contains α and has length $1/(k_i k_{i+1})$, so

$$\left|\alpha - \frac{h_i}{k_i}\right| < \frac{1}{k_i k_{i+1}} \leq \frac{1}{k_i(k_i+1)}, \quad i > 1.$$

This can be improved (Proposition 13.1.9 of Kumandure and Romero): For any i, at least one of the convergents h_i/k_i and h_{i+1}/k_{i+1} satisfies $|\alpha - h/k| < 1/(2k^2)$.

When $\alpha = \pi/2$, we have $\pi/2 = [a_0; a_1, a_2, \ldots] = [1; 1, 1, 3, 31, 1, 145, \ldots]$, whose first few convergents are

$$\frac{h_0}{k_0} = \frac{1}{1}, \quad \frac{h_1}{k_1} = \frac{2}{1}, \quad \frac{h_2}{k_2} = \frac{3}{2}, \quad \frac{h_3}{k_3} = \frac{11}{7}, \quad \frac{h_4}{k_4} = \frac{344}{219}, \quad \frac{h_5}{k_5} = \frac{355}{226}, \quad \ldots.$$

Claim 1. *If $i \geq 1$, k_i is odd, and $a_{i+1} \geq 2$, then $|\tan h_i| > h_i$. If, in addition, i is even, then* $\tan h_i > h_i$.

Proof of Claim 1. Write h/k for h_i/k_i. We have $|\pi/2 - h/k| < 1/(k_i k_{i+1})$, and $\pi/2 - h/k$ is positive when i is even. Therefore $|k\pi/2 - h| < 1/k_{i+1}$, so

$$\begin{aligned}|\tan h| &= |\tan(k\pi/2 - (k\pi/2 - h))| = |\cot(k\pi/2 - h)| > \cot(1/k_{i+1}) \\ &> k_{i+1} - 1/(2k_{i+1}) = a_{i+1}k_i + k_{i-1} - 1/(2k_{i+1}) > 2k = \big(2/(h/k)\big)h \geq h,\end{aligned}$$

where we have used the estimate $\cot\theta > (1/\theta) - (\theta/2)$, which is valid in the first quadrant. When i is even, the absolute value sign may be removed. □

Claim 2. *If $i \geq 3$ and both k_i and k_{i+1} are odd, then $|\tan h| > h$ holds for at least one of the two convergents $h/k \in \{h_i/k_i, h_{i+1}/k_{i+1}\}$.*

Proof of Claim 2. At least one of the convergents $h/k \in \{h_i/k_i, h_{i+1}/k_{i+1}\}$ satisfies $|\pi/2 - h/k| < 1/(2k^2)$, and hence $|k\pi/2 - h| < 1/(2k)$. Estimating $|\tan h|$ as in the proof of Claim 1, $|\tan h| = |\tan(k\pi/2 - (k\pi/2 - h))| = |\cot(k\pi/2 - h)| > \cot(1/(2k)) > 2k - 1/(2\cdot 2k) > 2k - 1 = (2/(h/k) - 1/h)\,h \geq (2/(11/7) - 1/11)\,h = (13/11)h > h$, which is valid for $i \geq 3$. □

(**a**) Let $S = \{i \mid k_i \text{ and } k_{i+1} \text{ are odd}\}$ and $T = \{i \mid k_i \text{ is odd and } a_{i+1} \geq 2\}$. The result follows from Claims 1 and 2 if we can show that $S \cup T$ is an infinite set. In fact we prove that $S \cup T$ meets every set of four consecutive positive integers.

Fix a positive integer i. At least one of k_i, k_{i+1} must be odd, and we replace i by $i+1$, if necessary, so that k_i is odd. If k_{i+1} is odd, then $i \in S$, we are finished. Otherwise, k_{i+1} is even, and then $k_{i+2} = a_{i+2}k_{i+1} + k_i$ is odd. If $i+2 \in T$, we are finished, so assume $i+2 \notin T$. This implies $a_{i+3} = 1$, and then $k_{i+3} = a_{i+3}k_{i+2} + k_{i+1} = k_{i+2} + k_{i+1}$ is odd. This last fact implies $i+2 \in S$.

(**b**) In view of Claim 1, we may assume that k_{2i} is odd for only finitely many integers i. Then k_{2i} is even and k_{2i+1} is odd for all sufficiently large i. Now $k_{2i+2} = a_{2i+2}k_{2i+1} + k_{2i}$, and so a_{2i} is even (and hence $a_{2i} \geq 2$) for all sufficiently large i. For fixed large i, set $h = h_{2i+1} + h_{2i}$ and $k = k_{2i+1} + k_{2i}$. Then $h_{2i}/k_{2i} < h/k < h_{2i+2}/k_{2i+2} < \pi/2 < h_{2i+1}/k_{2i+1}$. Since k is odd, $0 < \pi/2 - h/k < h_{2i+1}/k_{2i+1} - h/k = 1/(kk_{2i+1})$. Since $k_{2i+1} > k/2$,we have $0 < \pi/2 - h/k < 2/k^2$, so $0 < k\pi/2 - h < 2/k$. Therefore $\tan h = \tan(k\pi/2 - (k\pi/2 - h)) = \cot(k\pi/2 - h) > \cot(2/k) > k/2 - 1/k \geq (k-1)/2 = ((1/2)(k/h) - 1/(2h))\,h$. Since k/h is close to $2/\pi$ for large k, we have $(1/2)(k/h) - 1/(2h) \approx 1/\pi > 1/4$.

Editorial comment. Recent related problems from this MONTHLY include 10242 [1992, 675; 1997, 271] and 10640 [1998, 62]. The proposers remark: "Presumably for each $\alpha > 0$ there exist infinitely many positive n such that $\tan n > \alpha n$. This would be true if $\pi/2$ were a 'random' real number."

Solved also by J. Anglesio (France), R. Barbara (Lebanon), D. Callan, A. Stadler (Switzerland), A. Stenger, T. Trimble, C. Y. Yildirim (Turkey), SJSU Problems Ring, and the proposer.

The Ellipse in a Paper Cup

10664 [1998, 464]. *Proposed by Vasile A. Mihai, Toronto, Canada.* A paper cup in the shape of a right circular cone contains some water. Show that if one tips the cup at an angle θ without spilling the liquid, then the surface of the water describes an ellipse whose minor axis has length independent of θ.

Solution by J. Schaer, University of Calgary, Calgary, Canada. Let the cone be given by $z^2 = c(x^2 + y^2)$ and the initial water level by $z = h$. In this position, the surface is a circle of radius $b = h/\sqrt{c}$, and the volume is $V = \frac{\pi}{3}b^2h = \frac{\pi}{3}bA$, where A is the area of the "wet" triangle in the yz-plane. When the cone is tipped, the water surface is an ellipse with minor semiaxis b' and volume V'. We wish to show that if $V' = V$, then $b' = b$. In this case the converse is equivalent: It suffices to show that if $b' = b$, then $V' = V$. Rather than tipping the cone, we may consider cutting it by planes that are parallel to the x-axis and produce an ellipse with minor semiaxis b. Since this minor axis is parallel to the x-axis, the endpoints of the minor axis lie in the planes $x = \pm b$, and their projections into the yz-plane form a hyperbola H with equation $z^2 = c(b^2 + y^2)$. The asymptotes of H are the lines of intersection of the cone with the yz-plane. The major axis of the boundary ellipse lies in the yz-plane, its endpoints lie on the asymptotes of H, and its midpoint lies on H.

Proposition. *A segment that touches a given hyperbola at its midpoint and ends on the asymptotes of the hyperbola is tangent to the hyperbola, and the triangles formed by the asymptotes and such segments all have the same area.*

Proof. The described property of hyperbolas is invariant under affine transformations, and all hyperbolas are affinely equivalent to the hyperbola with equation $y = 1/x$. So it suffices to show the property for $y = 1/x$. This is a simple calculation. $\square$

Let h' be the height of the tipped cone whose base is the ellipse and whose vertex is 0, and let a be the major semiaxis. The Proposition implies that the area A' of the "wet" triangle is $ah' = A' = A = bh$. The volume of the tipped cone is therefore $V' = \frac{\pi}{3}bah' = \frac{\pi}{3}bA' = \frac{\pi}{3}bA = V$.

Editorial comment. This problem appeared earlier in this MONTHLY: In volume 19 (1912), it was proposed and solved by C. N. Schmall. For a related property of cones (which can be used to solve this problem) the reader is referred to R. J. Bagby, Volumes of Cones, this MONTHLY 103 (1996) 794-796.

Solved also by J. Anglesio (France), A. B. Ayoub, R. J. Bagby, M. Barra and C. Bernardi (Italy), M. Benedicty, G. D. Chakerian, R. J. Chapman (U. K.), J. Dou (Spain), J.-P. Grivaux (France), G. L. Isaacs, P. M. Jarvis and G. Atkins, W. Kim (South Korea), N. Lakshmanan, W. C. Lang, J. H. Lindsey II, J. Marengo, S. Metcalf, M. D. Meyerson, H. S. Morse, D. K. Nester, R. Patenaude, C. Popescu (Belgium), C. R. Pranesachar (India), C. Rosenkilde, A. Sasane (The Netherlands), L. Scribani (South Africa), P. Simeonov, W. R. Smythe, P. Szeptycki, L. Verriest, R. Voles (U. K.), Anchorage Math Solutions Group, Con Amore Problems Group (The Netherlands), GCHQ Problems Group (U. K.), and the proposer.