Editorial comment. Recent related problems from this MONTHLY include 10242 [1992, 675; 1997, 271] and 10640 [1998, 62]. The proposers remark: "Presumably for each $\alpha > 0$ there exist infinitely many positive n such that $\tan n > \alpha n$. This would be true if $\pi/2$ were a 'random' real number."

Solved also by J. Anglesio (France), R. Barbara (Lebanon), D. Callan, A. Stadler (Switzerland), A. Stenger, T. Trimble, C. Y. Yildirim (Turkey), SJSU Problems Ring, and the proposer.

The Ellipse in a Paper Cup

10664 [1998, 464]. *Proposed by Vasile A. Mihai, Toronto, Canada.* A paper cup in the shape of a right circular cone contains some water. Show that if one tips the cup at an angle θ without spilling the liquid, then the surface of the water describes an ellipse whose minor axis has length independent of θ.

Solution by J. Schaer, University of Calgary, Calgary, Canada. Let the cone be given by $z^2 = c(x^2 + y^2)$ and the initial water level by $z = h$. In this position, the surface is a circle of radius $b = h/\sqrt{c}$, and the volume is $V = \frac{\pi}{3}b^2h = \frac{\pi}{3}bA$, where A is the area of the "wet" triangle in the yz-plane. When the cone is tipped, the water surface is an ellipse with minor semiaxis b' and volume V'. We wish to show that if $V' = V$, then $b' = b$. In this case the converse is equivalent: It suffices to show that if $b' = b$, then $V' = V$. Rather than tipping the cone, we may consider cutting it by planes that are parallel to the x-axis and produce an ellipse with minor semiaxis b. Since this minor axis is parallel to the x-axis, the endpoints of the minor axis lie in the planes $x = \pm b$, and their projections into the yz-plane form a hyperbola H with equation $z^2 = c(b^2 + y^2)$. The asymptotes of H are the lines of intersection of the cone with the yz-plane. The major axis of the boundary ellipse lies in the yz-plane, its endpoints lie on the asymptotes of H, and its midpoint lies on H.

Proposition. *A segment that touches a given hyperbola at its midpoint and ends on the asymptotes of the hyperbola is tangent to the hyperbola, and the triangles formed by the asymptotes and such segments all have the same area.*

Proof. The described property of hyperbolas is invariant under affine transformations, and all hyperbolas are affinely equivalent to the hyperbola with equation $y = 1/x$. So it suffices to show the property for $y = 1/x$. This is a simple calculation. □

Let h' be the height of the tipped cone whose base is the ellipse and whose vertex is 0, and let a be the major semiaxis. The Proposition implies that the area A' of the "wet" triangle is $ah' = A' = A = bh$. The volume of the tipped cone is therefore $V' = \frac{\pi}{3}bah' = \frac{\pi}{3}bA' = \frac{\pi}{3}bA = V$.

Editorial comment. This problem appeared earlier in this MONTHLY: In volume 19 (1912), it was proposed and solved by C. N. Schmall. For a related property of cones (which can be used to solve this problem) the reader is referred to R. J. Bagby, Volumes of Cones, this MONTHLY 103 (1996) 794-796.

Solved also by J. Anglesio (France), A. B. Ayoub, R. J. Bagby, M. Barra and C. Bernardi (Italy), M. Benedicty, G. D. Chakerian, R. J. Chapman (U. K.), J. Dou (Spain), J.-P. Grivaux (France), G. L. Isaacs, P. M. Jarvis and G. Atkins, W. Kim (South Korea), N. Lakshmanan, W. C. Lang, J. H. Lindsey II, J. Marengo, S. Metcalf, M. D. Meyerson, H. S. Morse, D. K. Nester, R. Patenaude, C. Popescu (Belgium), C. R. Pranesachar (India), C. Rosenkilde, A. Sasane (The Netherlands), L. Scribani (South Africa), P. Simeonov, W. R. Smythe, P. Szeptycki, L. Verriest, R. Voles (U. K.), Anchorage Math Solutions Group, Con Amore Problems Group (The Netherlands), GCHQ Problems Group (U. K.), and the proposer.