REVIEWS

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The Four-Color Theorem. By Rudolf Fritsch and Gerda Fritsch, translated from the German by J. Peschke. Springer-Verlag, 1998, xvi + 260 pp., $29.95.

Reviewed by **John A. Koch**

This book "has been written to explain the Four Color Theorem to a lay readership," and, for the most part, it succeeds. The highest praise I can give such an effort is that I learned from it both bits of history and developments that have occurred since I was involved [**2**] in the solution of the problem in 1976. The book begins with a review of the historical foundations of the theorem and ends with a reference to a website [**5**] that displays the recent work of Robertson, Sanders, Seymour, and Thomas.

The Four Color Theorem has generated interest among mathematicians and non-mathematicians alike: "The regions of every planar map can be colored using no more than four colors such that those regions that are adjacent have different colors." Most amateur investigators immediately conjure up regions shaped like the spokes in a wheel. The requirement that adjacent regions touch at more than a single point is necessary for a meaningful theorem.

The historical section begins with the origin of the theorem in an observation of Francis Guthrie, whose younger brother Frederick submitted the problem to his professor Augustus de Morgan in 1852. Alfred Kempe appeared to have solved the problem in 1879 when he published his paper in the *American Journal of Mathematics Pure and Applied.* It is an interesting sidelight how Kempe, a lawyer and an Englishman, came to submit to this American publication, at the time a "comparatively insignificant" journal. In 1890, Percy Heawood identified an error in Kempe's proof. However, Kempe's arguments do yield a relatively simple proof of the Five Color Theorem.

The Fritsches particularly highlight the German connection to the theorem. The important efforts of Heinrich Heesch and Karl Durre led to Wolfgang Haken's involvement, and the interplay between these three and Ken Appel resulted in the unavoidable sets being winnowed down from one million elements to fewer than 2000. Most of the researchers in the Four Color field were aware of what the others were doing; I recall Appel relating that he and Haken stopped work on their approach in 1970 to investigate Shimamoto's supposed proof.

To prove the Four Color Theorem, one first translates it into an equivalent problem about graphs. The proof then breaks down into two major components: first the generation of an unavoidable set of configurations, and then the demonstration that no element of the unavoidable set can be in a minimal counterexample to the theorem.

One unavoidable configuration can easily be derived from Euler's formula relating the number of faces f, vertices v, and edges e of a graph:

$$v - e + f = 2. \tag{1}$$

Since each edge borders two faces, and each face is surrounded by at least three edges, it follows that $3f \leq 2e$, so that

$$f \leq 2e/3. \quad (2)$$

This inequality together with (1) yields $v - e + 2e/3 \geq 2$, which implies $3v - e \geq 6$, or

$$e \leq 3v - 6. \quad (3)$$

Consequently, there must be a vertex with degree less than or equal to 5 in a connected, planar graph with no self-loops. Indeed, suppose that all the vertices of a graph had degree greater than 5. Adding the degrees of the vertices would show that $2e \geq 6v$, or $e \geq 3v$, which would contradict (3). Thus, there must be a vertex of degree 1, 2, 3, 4, or 5 in any planar graph with no self-loops: this is the unavoidable set that Kempe used in his failed proof of the Four Color Theorem.

If there is a counterexample to the Four Color Theorem, then there is one with a minimal number of vertices, obviously at least five. The second part of the proof is to show that every element (called a configuration) of the unavoidable set is *reducible*, that is, cannot be in a minimal counterexample to the theorem.

Kempe attempted to show that a degree 5 vertex is reducible by using a process that became known as "Kempe chaining." The flaw Heawood noted was that Kempe changed the colors of two chains simultaneously.

The process of showing that a configuration f is reducible begins with assuming that f is embedded in a minimal counterexample to the Four Color Theorem. One removes f, yielding a smaller graph. Since the original graph was assumed to be a minimal counterexample, the smaller graph can be colored with four colors. Now replace f in the graph and try to extend the existing coloration of the ring surrounding f into the interior vertices. If this can be done for an arbitrary coloration of the ring, then f is called A-reducible.

Other types of reducible configurations allow one to examine fewer ring colorations: B-reductions involve merging ring vertices (thus causing their colors to be the same) or adding edges between ring vertices (thus causing their colors to be different), while C and D reductions involve replacing the original configuration with a configuration containing fewer vertices (so that the whole graph can be four colored) and examining the resulting possible ring colorations. Such reducers decrease the total possible number of ring colorations that must be examined. This becomes critical when one considers the combinatorial explosion in possible unique ring colorations:

ring size	colorations
10	2,461
11	7,381
12	22,144
13	66,430
14	199,291
15	597,872

After their historical discussion, the Fritsches begin with topological maps in Chapter 2. At the start of a section that proves lemmas concerning simple curves and the Jordan curve theorem, they state: "It must, however, be emphasized that many seemingly self-evident statements and theorems are sometimes difficult to prove rigorously." Chapter 3 provides the topological version of the Four Color Theorem. The terms regular map, vertex degree, circuit, and border vertex are defined, and lemmas are proved about the amusingly named "minimal criminal," which is a postulated minimal counterexample to the Four Color Theorem.

The authors take the combinatorial approach in Chapter 4, where they prove the duality of maps and graphs. The usual transformation is to consider a point as the capital of a region. These capital points (vertices) are joined by lines to capitals in adjacent countries. Thus, the problem becomes to color the vertices of a planar graph in such a way that adjacent vertices have different colors. This is the formulation of the problem that Appel, Haken, and I worked with most closely. The authors prove the Five Color Theorem in this chapter.

Chapter 5 discusses the combinatorics of the graphical version of the theorem. At the end of the chapter, the authors give the necessary definitions of reducible configuration and unavoidable set. Up to this point, they have proved most of the lemmas. In the remaining 70 pages, they describe the four types of reductions (A, B, C, and D) with detailed examples. They even list Durre's program written in Algol, with German comments.

The final 11 pages discuss general principles involved in the massive process of determining the unavoidable set. This process is described through obstructions in configurations, some "rules of thumb," and "geographical goodness."

An interesting aspect of the proof is that there is not a single unique unavoidable set. In fact, as the original proof developed, certain configurations that were found too difficult to reduce were replaced by others. The unavoidable set in the original paper consists of 1476 configurations. The proof of Robertson, Sanders, Seymour, and Thomas [**4**] uses 633 configurations, and it trims the number of discharging rules from more than 300 to only 32. Despite the improvement in the proof, it has not been reduced to a simple enough process to satisfy all mathematicians (or even all non-mathematicians). The proof still involves enough computer calculation that one cannot verify the result by hand.

The possibility of an error in the computer programs troubles some people. However, there are several parameters that come out of the reduction process, and others who have written programs to reduce configurations have achieved the same parameters for the same configurations. The situation is analogous to solving a riddle: once you know the answer, it seems trivial; but to find the solution may involve many exhaustive trials.

This book would be excellent for college students involved in topics courses or senior projects. The beginning basics are described in detail. Although there is a definite lack of information about the discharging procedures used to develop the unavoidable set, there is a useful bibliography and enough leads to keep good mathematics students busy.

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