NOTES

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The Hyperbolic Pythagorean Theorem in the Poincaré Disc Model of Hyperbolic Geometry

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Sometime in the sixth century B.C. Pythagoras of Samos discovered the theorem that now bears his name in Euclidean geometry. The extension of the Euclidean Pythagorean theorem to hyperbolic geometry, which is commonly known as the hyperbolic Pythagorean theorem (see [**3**, **5**, **6**, **9**–**11**]), does not have a form analogous to the Euclidean Pythagorean theorem, so some authors have concluded that a truly hyperbolic Pythagorean theorem does not exist. For example, Wallace and West assert "the Pythagorean theorem is strictly Euclidean" since "in the hyperbolic [Poincaré disc] model the Pythagorean theorem is not valid!" [**15**]. We show that a natural formulation of the hyperbolic Pythagorean theorem does exist: it expresses the square of the hyperbolic length of the hypotenuse of a hyperbolic right angled triangle as a natural "sum" of the squares of the hyperbolic lengths of the other two sides.

The most general Möbius transformation of the complex unit disc $D = \{z : |z| < 1\}$ in the complex z-plane [**2**, **4**, **8**],

$$z \mapsto e^{i\theta}\frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z), \tag{1}$$

defines the *Möbius addition* $\oplus$ in the disc, which allows the Möbius transformation of the disc to be viewed as a *Möbius left translation*

$$z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z_0 \in D$, and $\bar{z}_0$ is the complex conjugate of z_0. A left Möbius translation is also called a left *gyrotranslation* [**13**]. Left gyrotranslations occur frequently in hyperbolic geometry [**7**, p. 55]. and are sometimes called *hyperbolic pure translations* [**9**, p. 224].

The prefix *gyro* that we use to emphasize analogies stems from the *Thomas gyration*, which results, in turn, from the abstraction of the relativistic effect known as the *Thomas precession* [**13**, **14**]. The relevance of the Thomas precession to hyperbolic geometry is not unexpected [**9**, p. 251] since this geometry underlies relativistic velocities. The sensitivity of Thomas precession to the non-Euclidean nature of the geometry of spacetime has attracted NASA's interest in measuring the Thomas precession of gyroscopes of unprecedented accuracy in Earth orbit; see http://einstein.stanford.edu.

The *Poincaré hyperbolic distance function* in D is [**2**]

$$d(a,b) = \left| \frac{a-b}{1-\bar{a}b} \right| = |a \ominus b|, \tag{2}$$

where we use the obvious notation $a \ominus b = a \oplus (-b)$ for $a, b \in D$. It satisfies the Möbius triangle inequality

$$d(a,c) \leq d(a,b) \oplus d(b,c), \tag{3}$$

which involves the Möbius addition $\oplus$ of two real numbers in the complex unit disc D. We prove (3) after the proof of our main theorem and a discussion of some relevant group theoretic properties of Möbius addition. The right hand side of (3) can be written as

$$\tanh\left(\tanh^{-1} d(a,b) + \tanh^{-1} d(b,c)\right) \tag{4}$$

so that the Möbius triangle inequality can be written as an inequality

$$\tanh^{-1} d(a,c) \leq \tanh^{-1} d(a,b) + \tanh^{-1} d(b,c) \tag{5}$$

that involves the ordinary, rather than the Möbius, addition of real numbers. The hyperbolic distance function in D is commonly defined in the literature by [**7**, p. 53]

$$h(a,b) = \tanh^{-1} d(a,b) = \frac{1}{2} \ln \frac{1 + d(a,b)}{1 - d(a,b)} \tag{6}$$

rather than by $d(a,b)$ in which case we have in the triangle inequality

$$h(a,c) \leq h(a,b) + h(b,c) \tag{7}$$

for all $a, b, c \in D$. The complex unit disc with its Poincaré distance function, called the *Poincaré disc*, gives the Poincaré disc model of hyperbolic geometry, in which geodesic lines are circular arcs that intersect the boundary of the disc orthogonally [**3**].

Theorem. (The Hyperbolic Pythagorean Theorem) *Let* $\triangle abc$ *be a hyperbolic triangle in the Poincaré disc, whose vertices are the points* a, b *and* c *of the disc and whose sides (directed counterclockwise) are* $A = -b \oplus c$, $B = -c \oplus a$, *and* $C = -a \oplus b$. *If the two sides* A *and* B *are orthogonal, then* $|A|^2 \oplus |B|^2 = |C|^2$.

Proof: Let $\triangle abc$ be any hyperbolic triangle whose vertices are the points a, b, and c of the disc, and whose sides, A, B, and C, are geodesic segments that join the vertices, as shown in Figure 1. The measure of the hyperbolic angle between two sides of a hyperbolic triangle is given by the Euclidean measure of the angle formed by Euclidean tangent rays [**3**]. A *hyperbolic right triangle* is a hyperbolic triangle one of whose angles is $\pi/2$. Furthermore, let $\triangle abc$ be a hyperbolic right triangle whose sides A and B are orthogonal. Its right angle can be moved to the center of D by an appropriate Möbius transformation (1) such that its two orthogonal sides lie on the real and on the imaginary axes of D, as shown in Figure 1. Möbius transformations of the disc preserve both the hyperbolic length of geodesic segments and the measure of hyperbolic angles. Hence, the resulting triangle $\triangle a'b'c'$, obtained by moving $\triangle abc$ as shown in Figure 1, is congruent to $\triangle abc$ in the sense that the two triangles $\triangle a'b'c'$ and $\triangle abc$ possess equal hyperbolic lengths for corresponding sides and equal measures for corresponding angles.

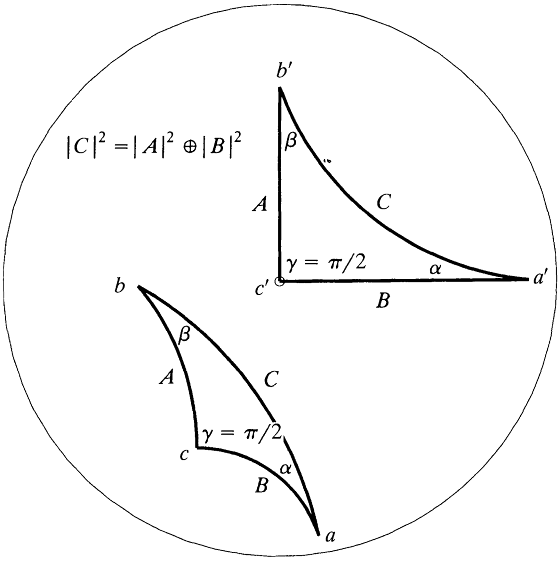

Figure 1. *The Hyperbolic Pythagorean Theorem in the complex unit disc.* The square of the hyperbolic length of the hypotenuse of a hyperbolic right triangle equals the Möbius sum of the squares of the hyperbolic lengths of the other two sides. Furthermore, $\sin\alpha = \gamma|A|/(\gamma_C|C|)$ and $\sin\beta = \gamma_B|B|/(\gamma_C|C|)$.

The vertices of the relocated hyperbolic right triangle $\triangle a'b'c'$ are $a' = x$, $b' = iy$, and $c' = 0$, for some $x, y \in (-1, 1)$. The hyperbolic length of the geodesic segment joining two points a and b of the disc is $d(a, b) = |b \ominus a|$. Accordingly, the hyperbolic lengths of the sides A, B, C of the triangle $\triangle a'b'c'$ are $|A|$, $|B|$, and $|C|$ given by

$$\begin{aligned} |A|^2 &= |b' \ominus c'|^2 = y^2, \\ |B|^2 &= |a' \ominus c'|^2 = x^2, \quad \text{and} \\ |C|^2 &= |a' \ominus b'|^2 = |x \ominus iy|^2 = \left|\frac{x - iy}{1 - ixy}\right|^2 = x^2 \oplus y^2. \end{aligned} \tag{8}$$

Hence

$$|A|^2 \oplus |B|^2 = |C|^2, \tag{9}$$

which verifies the hyperbolic Pythagorean theorem for hyperbolic right triangles in the Poincaré disc. ■

The Hyperbolic Pythagorean Theorem is not an isolated analogy with Euclidean geometry; analogies between the Poincaré disc model of hyperbolic geometry and Euclidean plane geometry abound in gyrogroup theory [**12**]. It is shown there that the Möbius addition, $\oplus$, is analogous to the common vector addition, $+$, in Euclidean plane geometry. If we define

$$\operatorname{gyr}[a; b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}, \tag{10}$$

then $\operatorname{gyr}[a;b]$ has modulus 1 and for all $a, b, c \in D$ the following group-like properties of $\oplus$ can be verified by straightforward algebra:

$a \oplus b = \operatorname{gyr}[a;b](b \oplus a)$	Gyrocommutative Law
$a \oplus (b \oplus c) = (a \oplus b) \oplus \operatorname{gyr}[a;b]c$	Left gyroassociative Law
$(a \oplus b) \oplus c = a \oplus (b \oplus \operatorname{gyr}[b;a]c)$	Right gyroassociative Law
$\operatorname{gyr}[a;b] = \operatorname{gyr}[a \oplus b; b]$	Left Loop Property
$\operatorname{gyr}[a;b] = \operatorname{gyr}[a; b \oplus a]$	Right Loop Property

A resulting geometrically important identity, also verifiable by straightforward algebra, is [**12**]

$$(x \oplus a) \ominus (x \oplus b) = \operatorname{gyr}[x, a](a \ominus b) \tag{11}$$

for all $a, b, x \in D$. Taking the modulus of each side of (11) gives

$$d(x \oplus a, x \oplus b) = d(a, b), \tag{12}$$

which shows that the Poincaré distance function (2) is invariant under Möbius left gyrotranslations.

To verify the Möbius triangle inequality (3), let $\gamma_a = (1 - |a|^2)^{-1/2}$ for any $a \in D$. Then $\gamma_a = \gamma_{|a|}$ is a monotonically increasing function of $|a|$ that satisfies the useful identity

$$\gamma_{a \oplus b} = \gamma_a \gamma_b |1 + \bar{a}b| \tag{13}$$

for all $a, b \in D$ [**1**, p. 2], as one can verify by squaring both sides.

It follows from (13) that

$$\gamma_{||a| \oplus |b||} = \gamma_{|a| \oplus |b|} = \gamma_{|a|}\gamma_{|b|}(1 + |a|\,|b|) \geq \gamma_a \gamma_b |1 + \bar{a}b| = \gamma_{a \oplus b} = \gamma_{|a \oplus b|}. \tag{14}$$

Since $||a| \oplus |b|| = |a| \oplus |b|$, and since $\gamma_z = \gamma_{|z|}$ is a monotonically increasing function of $|z|$, the inequality in (14) implies the inequality

$$|a| \oplus |b| \geq |a \oplus b| \tag{15}$$

for all $a, b \in D$.

Replacing x by $-x$ in (11), and noting that $-(-x \oplus b) = x \ominus b$, we have

$$(-x \oplus a) \oplus (x \ominus b) = \operatorname{gyr}[-x, a](a \ominus b) \tag{16}$$

for all $x, a, b \in D$. Finally, (16) and (15) imply

$$\begin{aligned} d(a,b) &= |a \ominus b| = |\operatorname{gyr}[-x, a](a \ominus b)| = |(-x \oplus a) \oplus (x \ominus b)| \\ &\leq |-x \oplus a| \oplus |x \ominus b| = d(a, x) \oplus d(x, b) \end{aligned}$$

for all $a, b, x \in D$, which proves the Möbius triangle inequality (3).

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Is the Composite Function Integrable?

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It is well known that the composition of two continuous functions is continuous and hence Riemann integrable. However, the composition of two Riemann integrable functions may or may not be Riemann integrable. For example, let

$$f(y) = \begin{cases} 1 & \text{when } y \neq 0, \\ 0 & \text{when } y = 0, \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{when } x \text{ is an irrational number,} \\ \frac{1}{p} & \text{when } x = \frac{q}{p}, \text{ where } p \text{ and } q \text{ are two coprime integers.} \end{cases}$$

Then

$$f \circ g(x) = \begin{cases} 0 & \text{when } x \text{ is an irrational number,} \\ 1 & \text{when } x = \frac{q}{p}, \text{ where } p \text{ and } q \text{ are two coprime integers.} \end{cases}$$

Both f and g are Riemann integrable on $[0, 1]$, but the composition $f \circ g$ is not. Therefore, it is natural to ask whether the composition of two functions is still Riemann integrable, when one is Riemann integrable and the other is continuous.

In what follows, we let f be a function defined on the interval $[a, b]$, and let g be a function defined on the interval $[c, d]$ with its range contained in $[a, b]$.

Question 1. If f is continuous on $[a, b]$ and g is Riemann integrable on $[c, d]$, is the composition $f \circ g$ Riemann integrable on $[c, d]$?

The answer is yes.