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Rationals and the Modular Group

Roger C. Alperin

The modular group \mathcal{M} is the quotient group $PSL_2(\mathbf{Z}) = SL_2(\mathbf{Z})/\{\pm I\}$ of $SL_2(\mathbf{Z})$, the group of 2×2 integer matrices of determinant 1. In [1] we gave an elementary proof that \mathcal{M} has the structure of a free product of a cyclic group of order 2 generated by the image of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and a cyclic group of order 3 generated by the image of $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

The free product structure provides a description of the non-trivial elements of \mathcal{M} as unique strings of A 's and B 's with the property that there are no two consecutive A 's and no three consecutive B 's; we refer to these as *reduced strings*. We explained this free product structure in terms of the action of the modular group on the irrationals. In this note we describe the action on the rationals; this can be viewed as a way of describing the inverse of the Euclidean algorithm.

The group $SL_2(\mathbf{Z})$ acts via linear transformations on \mathbf{R}^2 as column vectors and this gives an action of \mathcal{M} via linear fractional transformations on the projective line $P^1(\mathbf{R})$, the real numbers together with ∞ . We may also view $P^1(\mathbf{R})$ as the slopes of non-zero vectors, that is, the equivalence classes of $\mathbf{R}^2 - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ induced by non-zero scalar multiplication; the equivalence class of the vector $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is denoted ∞ , the equivalence class of the vector $\begin{pmatrix} p \\ q \end{pmatrix}$, $q \neq 0$ is the same as that of $\begin{pmatrix} p/q \\ 1 \end{pmatrix}$ and corresponds to the real number $z = p/q$. For the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbf{Z})$ the induced action on $P^1(\mathbf{R})$ is given by

$$z \rightarrow \frac{az + b}{cz + d}.$$

The induced action for the generating elements is given as

$$A: z \rightarrow \frac{-1}{z}, \quad B: z \rightarrow \frac{-1}{z+1}, \quad B^2: z \rightarrow -1 - \frac{1}{z}.$$

The orbit of e is easily seen to be in correspondence with the set of all first columns of matrices from $SL_2(\mathbf{Z})$. Thus the orbit of ∞ is in 1-1 correspondence with the projective line $P^1(\mathbf{Q})$, consisting of the set \mathbf{Q} of all reduced fractions together with ∞ ; from elementary group theory this is in 1-1 correspondence with the set of left cosets of the stabilizer \mathcal{N} of e , which is the image of the subgroup generated by AB in \mathcal{M} .

Using the free product description of \mathcal{M} we can also describe the set of coset representatives as reduced strings of A 's and B 's. First, a non-trivial coset representative cannot end in AB or its inverse B^2A ; therefore if it ends in A it is either A or of the form ZBA with Z ending in A or trivial; if it ends in B it is either B or ZB^2 with Z ending in A or trivial. Thus, as a first pass, the set of coset representatives is the set $\mathcal{R} = \{I\} \cup \{A\} \cup \{B\} \cup \{BA\} \cup \{B^2\} \cup \{ZBA | Z \text{ any string ending in } A\} \cup \{ZB^2 | Z \text{ any string ending in } A\}$. Next, to determine the distinct coset representatives we just observe that the free product description

gives a unique expression for the elements. The coset equivalence relation $X \cong Y$ on reduced strings $X, Y \in \mathcal{M}$ is $Y = X(AB)^n$ or $Y = X(B^2A)^n$ for some non-negative integer n . We see easily that $A \cong B \pmod{\mathcal{N}}$ and hence also $XA \cong XB \pmod{\mathcal{N}}$ for any string X , and hence if this is reduced, $X \neq I$ must end in B . Thus we can simplify the description of the distinct coset representatives to $\mathcal{R} = \{I\} \cup \{B\} \cup \{B^2\} \cup \{ZB^2 \mid Z \text{ any string ending in } A\}$. It is easy to see that no two of these reduced strings are equivalent; for example, for $Z \neq W$, both ending in A , then $ZB^2 = WB^2(B^2A)^n$ and $ZB^2 = WB^2(AB)^n$ are impossible. Thus the coset representatives of \mathcal{M}/\mathcal{N} are the distinct strings $\mathcal{R} = \{I\} \cup \{B\} \cup \{B^2\} \cup \{ZB^2 \mid Z \text{ any string ending in } A\}$.

We can also describe this set \mathcal{R} as the union of \mathcal{R}_m defined inductively as

$$\begin{aligned} \mathcal{R}_0 &= \{I, B\}, \quad \mathcal{N}_0 = \{B^2\} \\ \mathcal{P}_m &= \{A\}\mathcal{N}_m, \quad \mathcal{N}_{m+1} = \{B^2, B\}\mathcal{P}_m \\ \mathcal{R}_{m+1} &= \mathcal{R}_m \cup \mathcal{P}_m \cup \mathcal{N}_m \end{aligned} \tag{1}$$

It is easy to see that \mathcal{R}_m has 2^{m+1} elements and $\mathcal{P}_m, \mathcal{N}_m$ each have 2^m elements. We can rewrite (1) as

$$\mathcal{P}_{m+1} = \{AB, AB^2\}\mathcal{P}_m, \quad \mathcal{N}_{m+1} = \{B^2A, BA\}\mathcal{N}_m. \tag{2}$$

Simplifying (2) we obtain the following result. $\mathcal{FS}(x, y)$ denotes the free semigroup with the generators, x, y .

Proposition. $P^1(\mathbf{Q})$ is in 1-1 correspondence with $\mathcal{R} = \{I\} \cup \{B\} \cup \mathcal{FS}(AB, AB^2) \cdot AB^2 \cup \mathcal{FS}(B^2A, BA) \cdot B^2$.

Observing, that $I_\infty = \infty, B_\infty = 0, B^2_\infty = -1$ and $AB^2_\infty = 1$, and $P^1(\mathbf{Q}) - \{\infty, 0\}$ is the positive and negative rationals, we have the following

Corollary. *The set of positive rationals is the orbit of the free semigroup generated by AB and AB^2 on $z = 1$. The set of negative rationals is the orbit of the free semigroup generated by B^2A and BA on $z = -1$.*

These upper $U = AB, U^- = B^2A$ and lower $L = AB^2, L^- = BA$ triangular matrix actions corresponding to these semigroup generators are

$$\begin{aligned} U: z &\rightarrow z + 1, \quad L: z \rightarrow \frac{z}{z + 1}, \\ U^-: z &\rightarrow z - 1, \quad L^-: z \rightarrow \frac{z}{1 - z}. \end{aligned}$$

Every positive rational is uniquely expressible in terms of semigroup generators as an element of the orbit of 1. Alternatively, starting from a reduced positive

rational $z = \frac{p}{q}$ we can apply a greedy or Euclidean recursion to obtain a finite sequence that stabilizes at 1;

$$\frac{p}{q} \rightarrow \begin{cases} \frac{(p-q)}{q} & \text{if } p > q \\ \frac{p}{(q-p)} & \text{if } p < q \\ 1 & \text{if } p = q = 1 \end{cases}$$

Here we apply U^- or L^- depending on whether or not $z > 1$ or $z < 1$. For example, the sequence

$$\frac{34}{55}, \frac{34}{21}, \frac{13}{21}, \frac{13}{8}, \frac{5}{8}, \frac{5}{3}, \frac{2}{3}, \frac{2}{1}, 1$$

corresponds to $(U^-L^-)^4(\frac{34}{55}) = 1$. Since we have shown that the set of positive rationals can be described by a free semigroup, this means that $(LU)^4L$ is the coset representative in \mathcal{R}_9 corresponding to $34/55$ as described in the Corollary.

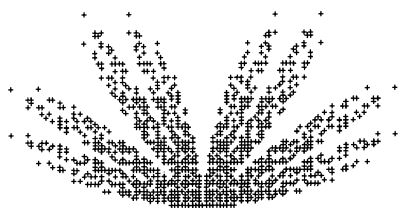


Figure 1. \mathcal{R}_9 .

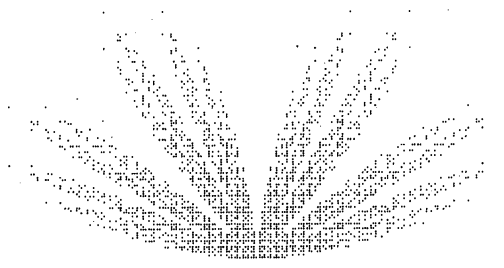


Figure 2. \mathcal{R}_{10} .

Finally, we consider the matrix action of the distinct non-trivial coset representatives in \mathcal{R} on the column $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the plots of these images in \mathbf{R}^2 . These images are just the points with relatively prime coordinates in the upper half-plane. We obtain the fascinating plant-like structures in Figures 1 and 2. For example, the distant points from the root $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the ‘Fibonacci points’ $\begin{pmatrix} \pm 34 \\ 55 \end{pmatrix}$, $\begin{pmatrix} \pm 55 \\ 34 \end{pmatrix}$ in \mathcal{R}_9 .

REFERENCE

1. Roger C. Alperin, $PSL_2(\mathbf{Z}) = \mathbf{Z}_2 * \mathbf{Z}_3$ *Amer. Math. Monthly* **100** (1993) 385–386.

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