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Field Theory: From Equations to Axiomatization

Part II

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7. THE ABSTRACT DEFINITION OF A FIELD. The developments we have been describing thus far lasted close to a century. They gave rise to important "concrete" theories—Galois theory, algebraic number theory, algebraic geometry—in which the (at times implicit) field concept played a central role. At the end of the 19th century abstraction and axiomatics were "in the air." For example, Pasch (1882) gave axioms for projective geometry, stressing for the first time the importance of undefined notions, Cantor (1883) defined the real numbers essentially as equivalence classes of Cauchy sequences of rationals, and Peano (1889) gave his axioms for the natural numbers. In algebra, von Dyck (1882) gave an abstract definition of a group that encompassed both finite and infinite groups (about thirty years earlier Cayley had defined a *finite* group), and Peano (1888) gave a definition of a finite-dimensional vector space, though this was largely ignored by his contemporaries. The time was propitious for the abstract field concept to emerge. Emerge it did in 1893 in the hands of Weber (of Dedekind-Weber fame).

Weber's definition of a field appeared in his 1893 paper "Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie" [**15**], in which he aimed to give an abstract formulation of Galois theory [**8**, p. 136]:

> In the following an attempt is made to present the Galois theory of algebraic equations in a way which will include equally well all cases in which this theory might be used. Thus we present it here as a direct consequence of the group concept illuminated by the field concept, as a formal structure completely without reference to any numerical interpretation of the elements used.

Weber's presentation of Galois theory is indeed very close to the way the subject is taught today. His definition of a field, preceded by that of a group, is as follows [**15**, pp. 526–527]:

> A group becomes a field if two types of composition are possible in it, the first of which may be called *addition*, the second *multiplication*. The general determination must be somewhat restricted, however.
> 1. We assume that both types of composition are commutative.
> 2. Addition shall generally satisfy the conditions which define a group.

3. Multiplication is such that

$a(-b) = -(ab)$

$a(b + c) = ab + ac$

$ab = ac$ implies $b = c$, unless $a = 0$

Given b and c, $ab = c$ determines a, unless $b = 0$.

Although the associative law under multiplication is missing, and the axioms are not independent, they are of course very much in the modern spirit. As examples of his newly defined concept Weber included the number fields and function fields of algebraic number theory and algebraic geometry, respectively, but also Galois's finite fields and Kronecker's "congruence fields" $K[x]/(p(x))$, K a field, $p(x)$ irreducible over K.

Weber proved (often reproved, after Dedekind) various theorems about fields, which later became useful in Artin's formulation of Galois theory, and which are today recognized as basic results of the theory. Among them are [**8**], [**10**]:

(i) Every finite algebraic extension of a field is simple (that is, it is generated by a single element).
(ii) Every polynomial over a field has a splitting field.
(iii) If $F \subseteq F(a) \subseteq F(b)$, then $(F(a): F)$ divides $(F(b): F)$, where for fields K and E with $E \subseteq K$, $(K : E)$ denotes the dimension of K as a vector space over E.

It should be emphasized that it was not Weber's aim to study fields as such, but rather to develop enough of field theory to give an abstract formulation of Galois theory [**11**]. In this he succeeded admirably. His paper, and somewhat later his two-volume *Lehrbuch der Algebra*, exerted considerable influence on the development of abstract algebra [**3**].

8. HENSEL'S *P*-ADIC NUMBERS. In an 1899 article entitled "New foundations of the theory of algebraic numbers," Hensel began a life-long study of p-adic numbers. Inspired by the work of Dedekind-Weber, Hensel took as his point of departure the analogy between function fields and number fields. Just as power series are useful for a study of the former, Hensel introduced p-adic numbers to aid in the study of the latter [**10**, II, p. 19]:

> The analogy between the results of the theory of algebraic functions of one variable and those of the theory of algebraic numbers suggested to me many years ago the idea of replacing the decomposition of algebraic numbers, with the help of ideal prime factors, by a more convenient procedure that fully corresponds to the expansion of an algebraic function in power series in the neighborhood of an arbitrary point.

Indeed, in the neighborhood of a given point α every algebraic function of a complex variable can be represented as an infinite series of integral and rational powers of $z - \alpha$, as Weierstrass had shown. The elements of Hensel's *field of p-adic numbers* are formal power series $\Sigma_n^\infty a_k p^k$, where $a_k \in Z_p$ and $n \in Z$. And just as every element of an algebraic function field can be identified with the set of its expansions at all points of the Riemann surface on which it is defined, so every element of an algebraic number field is identified with the set of its representations in the field of p-adic numbers $\Sigma_n^\infty a_k p^k$ for every prime p [**2**, p. 111].

In a 1907 book, Hensel introduced topological notions in his p-adic fields and applied the resulting p-adic analysis in algebraic number theory. The p-adic numbers proved extremely useful not only there but also in algebraic geometry [**4**], [**7**]. They were also influential in motivating the abstract study of rings and fields [**3**].

9. STEINITZ. The last major event in the evolution of field theory that we describe is Steinitz's great work of 1910 [**13**]. But first some background.

Algebra in the 19th century was by our standards concrete. It was connected in one way or another with the real or complex numbers. For example, some of the great contributors to 19th-century algebra, mathematicians whose ideas shaped the algebra of the 20th century, were Gauss, Galois, Jordan, Kronecker, Dedekind, and Hilbert, and their algebraic work dealt with quadratic forms, cyclotomy, permutation groups, ideals in rings of algebraic number fields and algebraic function fields, and invariant theory. All of these subjects were related in one way or another to the real or complex numbers.

At the turn of the 20th century the axiomatic method began to take hold as an important mathematical tool. Hilbert's *Foundations of Geometry* of 1899 was very influential in this respect (see also our Section 7). Noteworthy also is the American school of axiomatic analysis, as exemplified in the works of Dickson, Huntington, E. H. Moore, and Veblen. In the first decade of the 20th century these mathematicians began to examine various axiom systems for groups, fields, associative algebras, projective geometry, and the algebra of logic. Their principal aim was to study the independence, consistency, and completeness of the axioms defining any one of these systems. Also relevant were Hilbert's axiomatic characterization in 1900 of the field of real numbers and Huntington's like characterization in 1905 of the field of complex numbers [**1**], [**3**].

Steinitz's groundbreaking 150-page paper "Algebraische Theorie der Körper" of 1910 initiated the abstract study of fields as an independent subject [**13**]. While Weber *defined* fields abstractly, Steinitz *studied* them abstractly.

Steinitz's immediate source of inspiration was Hensel's p-adic numbers [**3**, p. 194]:

> I was led into this general research especially by Hensel's *Theory of Algebraic Numbers*, whose starting point is the field of p-adic numbers, a field which counts neither as a field of functions nor as a field of numbers in the usual sense of the word.

More generally, Steinitz's work arose out of a desire to delineate the abstract notions common to the various contemporary theories of fields: fields in algebraic number theory, in algebraic geometry, and in Galois theory, p-adic fields, and finite fields. His goal was a comprehensive study of *all* fields, starting from the field axioms [**3**, p. 195]:

> The aim of the present work is to advance an overview of all the possible types of fields and to establish the basic elements of their interrelations.

Quite a task! Steinitz's plan was to start from the simplest fields and to build up all fields from these. The basic concept that he identified to study the former is the *characteristic* of the field. Here are several of his fundamental results, nowadays staples of field theory [**10**], [**13**]:

(i) Classification of fields into those of characteristic zero and those of characteristic p. The *prime fields*—the "simplest" fields—are Q and Z_p; one or the other is a subfield of every field.
(ii) Development of a theory of *transcendental extensions*, which became indispensable in algebraic geometry.
(iii) Recognition that it is precisely the *finite*, *normal*, *separable extensions* to which Galois theory applies.
(iv) Proof of the existence and uniqueness (up to isomorphism) of the *algebraic closure* of any field.

A description of all fields followed [**11**, p. 754]:

> Starting with an arbitrary prime field, by taking an arbitrary, purely transcendental extension followed by an arbitrary algebraic extension, we have a method of arriving at any field.

The notions of *transcendency base* and *degree of transcendence* of an extension field, both of which Steinitz introduced, played a crucial role here. Also important was the well-ordering principle (or, equivalently, the axiom of choice), whose use he acknowledged [**10**, II, p. 20]:

> Many mathematicians continue to reject the axiom of choice. The growing realization that there are questions in mathematics that cannot be decided without this principle is likely to result in the gradual disappearance of the resistance to it.

Steinitz's work was very influential in the development of abstract algebra in the 1920s and 1930s, as the following testimonials prove:

> Steinitz's paper was the basis for all [algebraic] investigations in the school of Emmy Noether (van der Waerden, [**14**, p. 162]).
>
> [Steinitz's work]... is not only a landmark in the development of algebra, but also... an excellent, in fact indispensable, introduction to a serious study of the new [modern] algebra (Baer and Hasse, [**13**, Preface]).
>
> Steinitz's work marks a methodological turning-point in algebra leading to ...'modern' or abstract algebra (Purkert and Wussing, [**11**, p. 754]).
>
> [Steinitz's work] can be considered as having given birth to the actual concept of Algebra (Bourbaki, [**2**, p. 83]).

10. A GLANCE AHEAD. We now list several major developments in field theory and related areas in the decades following Steinitz's fundamental work.

(a) *Valuation theory*. In 1913 Kürschak abstracted Hensel's ideas on p-adic fields by introducing the notion of a *valuation field*. He proved the existence of the completion of a field with respect to a valuation. In 1918 Ostrowski determined all valuations of the field Q of rational numbers. Valuation theory, which "forms a solid link between number theory, algebra, and analysis" [**7**, vol. II, p. 537], played fundamental roles in both algebraic number theory and algebraic geometry; see [**2**], [**4**], [**7**], [**14**].

(b) *Formally real fields*. In 1927 Artin and Schreier defined the notion of a *formally real field*, namely a field in which -1 is not a sum of squares. "One of [the] remarkable results [of the Artin-Schreier theory] is no doubt the discovery that the existence of an order relation on a field is linked to purely algebraic properties of the field" [**2**, p. 92]: A field can be ordered if and only if it is formally real. The theory of formally real fields enabled Artin in the same year to solve *Hilbert's 17th Problem* on the resolution of positive definite rational functions into sums of squares [**7**, vol. II, p. 640].

(c) *Class field theory*. This is the study of finite extensions of an algebraic number field having an abelian Galois group. It is a beautiful synthesis of algebraic, number-theoretic, and analytic ideas, in which *Artin's Reciprocity Law* has a central place. Major strides were already made by Hilbert in his "Zahlbericht" (Report on Number Theory) of 1897. More modern aspects of the theory were developed by Artin, Chevalley, Hasse, Tagaki, and others; see [**5**].

(d) *Galois theory*. Artin set out his now-famous abstract formulation of Galois theory in lectures given in 1926 (but published only in 1938). In a 1950 talk he said [**8**, p. 144]:

> Since my mathematical youth I have been under the spell of the classical theory of Galois. This charm has forced me to return to it again and again, and try to find new ways to prove its fundamental theorems.

Extensions of the classical theory were given in various directions. For example, in 1927 Krull developed a Galois theory of *infinite field extensions*, establishing a one-one correspondence between subfields and "closed" subgroups, and thereby introducing topological notions into the theory. There is also a Galois theory for *inseparable field extensions*, in which the notion of derivation of a field plays a central role, and a Galois theory for *division rings*, developed independently by H. Cartan and Jacobson in the 1940s; see [**7**], [**16**].

(e) *Finite fields*. Finite field theory is a thriving subject of investigation in its own right, but it also has important uses in number theory, coding theory, geometry, and combinatorics; see [**6**], [**9**].

REFERENCES

1. G. Birkhoff, Current trends in algebra, *Amer. Math. Monthly* **80** (1973) 760–782, and corrections in **81** (1974) 746.
2. N. Bourbaki, *Elements of the History of Mathematics*, Springer-Verlag, 1984.
3. L. Corry, *Modern Algebra and the Rise of Mathematical Structures*, Birkhäuser, 1996.
4. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 1995.
5. H. Hasse, History of class field theory, in *Algebraic Number Theory, Proceedings of an Instructional Conference*, ed. by J. Cassels and A. Fröhlich, Thompson Book Co., 1967, pp. 266–279.
6. K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer-Verlag, 1982.
7. N. Jacobson, *Basic Algebra I, II*, W. H. Freeman, 1974 and 1980.
8. B. M. Kiernan, The development of Galois theory from Lagrange to Artin, *Arch. Hist. Exact Sci.* **8** (1971/72) 40–154.
9. R. Lidl and H. Niederreiter, *Introduction to Finite Fields and their Applications*, Cambridge University Press, 1986.
10. W. Purkert, Zur Genesis des abstrakten Körperbegriffs I, II, *Naturwiss., Techn. u. Med.* **8** (1971) 23–37 and **10** (1973) 8–20. Unpublished English translation by A. Shenitzer.
11. W. Purkert and H. Wussing, Abstract algebra, in *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, vol. I, ed. by I. Grattan-Guinness, Routledge, 1994, pp. 741–760.
12. J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer-Verlag, 1992.
13. E. Steinitz, *Algebraische Theorie der Körper*, 2nd ed., Chelsea, 1950.
14. B. L. van der Waerden, Die Algebra seit Galois, *Jahresber. Deutsch. Math. Verein.* **68** (1966) 155–165.
15. H. Weber, Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie, *Math. Ann.* **43** (1893) 521–549.
16. D. Winter, *The Structure of Fields*, Springer-Verlag, 1974.