PROBLEMS AND SOLUTIONS

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before April 30, 2000; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10760. *Proposed by Bruce Reznick, University of Illinois, Urbana, IL.* A function $f\colon \mathbb{N} \to \mathbb{C}$ is *completely multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and n. Find all completely multiplicative functions f with the property that the function $F(n) = \sum_{k=1}^{n} f(k)$ is also completely multiplicative.

10761. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let G be a graph with n vertices. For each vertex v, let $f(v)$ be the maximum cardinality of an independent set of neighbors of v. Show that $\sum f(v) \le n^2/2$, where the sum is taken over all vertices of G.

10762. *Proposed by Leroy Quet, Denver, CO.* Let $x_1 = 1$, and for $m \ge 1$ let $x_{m+1} = (m + 3/2)^{-1} \sum_{k=1}^{m} x_k x_{m+1-k}$. Evaluate $\lim_{m\to\infty} x_m/x_{m+1}$.

10763. *Proposed by Jean Anglesio, Garches, France.* Let ABC be a triangle; let O be its circumcenter, H its orthocenter, I its incenter, N its Nagel point, and X, Y, Z its excenters. Let S be defined so that O is the midpoint of HS, and let T denote the midpoint of SN. It is known that the orthocenter and the nine-point center of triangle XYZ are I and O, respectively. Prove that
(a) the circumcenter of triangle XYZ is T; and
(b) the centroid of triangle XYZ is the centroid of SIN.

10764. *Proposed by Ray Redheffer, University of California, Los Angeles, CA.* Let $A = (a_{ij})$ be a real n-by-n matrix, and let x and y be real n-vectors satisfying $Ax = y$. Suppose that

$$\sum_{j\ne i} \max\{a_{ij}, 0\} < y_i \le a_{ii} + \sum_{j\ne i} \min\{a_{ij}, 0\}$$

for all $i \in \{1, 2, \ldots, n\}$. Show that $x_i > 0$ for all $i \in \{1, 2, \ldots, n\}$.

10765. *Proposed by Peter J. Ferraro, Roselle Park, NJ.* Let f_n be the nth Fibonacci number, defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \ge 1$. Fix positive integers k and n with $n \ge 2k + 1$. Prove that $\lfloor \sqrt[k]{f_n} \rfloor - \lfloor \sqrt[k]{f_{n-k}} + \sqrt[k]{f_{n-2k}} \rfloor$ is 0 unless f_n is a kth power, when it is 1.