

10766



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The American Mathematical Monthly, Vol. 106, No. 9. (Nov., 1999), p. 865.

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10766. Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania. Let x , y , and z be nonnegative real numbers. Prove that

(a) $(x + y + z)^{x+y+z} x^x y^y z^z \leq (x + y)^{x+y} (y + z)^{y+z} (z + x)^{z+x}$.

(b) $(x + y + z)^{(x+y+z)^2} x^{x^2} y^{y^2} z^{z^2} \geq (x + y)^{(x+y)^2} (y + z)^{(y+z)^2} (z + x)^{(z+x)^2}$.

SOLUTIONS

Cramer's Rule for Non-Square Matrices

10618 [1997, 768]. Proposed by S. Lakshminarayanan, S. L. Shah, and K. Nandakumar, University of Alberta, Edmonton, Canada. Let A be a real $m \times n$ matrix of full rank with $m < n$ and let b be a real $m \times 1$ matrix. For $1 \leq i \leq n$, define

$$x_i = \frac{\det(A_i^* A^T) - \det(A_i A_i^T)}{\det(AA^T)},$$

where A_i^* is obtained by replacing the i th column of A by b , and A_i is obtained by deleting the i th column of A . Show that $x = [x_1, \dots, x_n]^T$ is a solution to the linear system $Ax = b$.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We write $A^i \langle b \rangle$ instead of A_i^* to emphasize the role of the vector b ; thus $A^i \langle 0 \rangle$ indicates A with its i th column zeroed out. Observe that $A_i A_i^T = A^i \langle 0 \rangle A^T$, by comparing corresponding entries.

Extend A to a nonsingular $n \times n$ matrix $\begin{pmatrix} A \\ C \end{pmatrix}$, where C is an $(n - m) \times n$ matrix whose rows form an orthonormal basis for the orthogonal complement of the row space of A . That is, each row of C has norm 1 and is orthogonal to all other rows of $\begin{pmatrix} A \\ C \end{pmatrix}$. We have

$$\begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} AA^T & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^i \langle b \rangle \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} A^i \langle b \rangle A^T & M \\ 0 & I \end{pmatrix},$$

where I is the $(n - m) \times (n - m)$ identity matrix and M is some $n \times (n - m)$ matrix. By substituting these computations into the definition of x_i , canceling the nonzero factor $\det \begin{pmatrix} A \\ C \end{pmatrix}^T$, and using the linearity of the determinant in its i th column, we obtain

$$x_i = \frac{\det \left(\begin{pmatrix} A^i \langle b \rangle \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T \right) - \det \left(\begin{pmatrix} A^i \langle 0 \rangle \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T \right)}{\det \left(\begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T \right)} = \frac{\det \begin{pmatrix} A^i \langle b \rangle \\ C \end{pmatrix} - \det \begin{pmatrix} A^i \langle 0 \rangle \\ C \end{pmatrix}}{\det \begin{pmatrix} A \\ C \end{pmatrix}} = \frac{\det \begin{pmatrix} A \\ C \end{pmatrix}^i \frac{b}{0}}{\det \begin{pmatrix} A \\ C \end{pmatrix}},$$

By Cramer's rule, x is the solution to the linear system $\begin{pmatrix} A \\ C \end{pmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix}$, and hence x is a solution to $Ax = b$.

Solved also by J. Fuelberth & A. Gunawardena, J. H. Lindsey II, M. Sharma & P. G. Poonacha (India), WMC Problems Group, and the proposers.

An Identity for Strongly Connected Digraphs

10620 [1997, 870]. Proposed by James Propp, Massachusetts Institute of Technology, Cambridge, MA. A digraph on a vertex set V is a subset $A \subseteq \{(v, w) : v, w \in V, v \neq w\}$ and is *strongly connected* if it is possible to get from any vertex a to every other vertex e by a finite succession of arcs $(a, b), (b, c), \dots, (d, e)$ in A . For $n \geq 1$, let E_n (respectively, O_n) denote the number of strongly connected digraphs on the vertex set $V = \{1, 2, \dots, n\}$ with an even (respectively odd) number of arcs. Show that $E_n - O_n = (n - 1)!$ for all $n \geq 1$.

Solution I by the proposer, currently at University of Wisconsin, Madison, WI. The terminology of the problem statement is somewhat nonstandard. In common usage, a digraph is