



## Cramer's Rule for Non-Square Matrices: 10618

S. Lakshminarayanan; S. L. Shah; K. Nandakumar; GCHQ Problems Group

*The American Mathematical Monthly*, Vol. 106, No. 9. (Nov., 1999), p. 865.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9890%28199911%29106%3A9%3C865%3ACRFNM1%3E2.0.CO%3B2-J>

*The American Mathematical Monthly* is currently published by Mathematical Association of America.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/maa.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

**10766.** Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania. Let  $x$ ,  $y$ , and  $z$  be nonnegative real numbers. Prove that

(a)  $(x + y + z)^{x+y+z} x^x y^y z^z \leq (x + y)^{x+y} (y + z)^{y+z} (z + x)^{z+x}$ .

(b)  $(x + y + z)^{(x+y+z)^2} x^{x^2} y^{y^2} z^{z^2} \geq (x + y)^{(x+y)^2} (y + z)^{(y+z)^2} (z + x)^{(z+x)^2}$ .

## SOLUTIONS

### Cramer's Rule for Non-Square Matrices

**10618** [1997, 768]. Proposed by S. Lakshminarayanan, S. L. Shah, and K. Nandakumar, University of Alberta, Edmonton, Canada. Let  $A$  be a real  $m \times n$  matrix of full rank with  $m < n$  and let  $b$  be a real  $m \times 1$  matrix. For  $1 \leq i \leq n$ , define

$$x_i = \frac{\det(A_i^* A^T) - \det(A_i A_i^T)}{\det(AA^T)},$$

where  $A_i^*$  is obtained by replacing the  $i$ th column of  $A$  by  $b$ , and  $A_i$  is obtained by deleting the  $i$ th column of  $A$ . Show that  $x = [x_1, \dots, x_n]^T$  is a solution to the linear system  $Ax = b$ .

*Solution by the GCHQ Problems Group, Cheltenham, U. K.* We write  $A^i \langle b \rangle$  instead of  $A_i^*$  to emphasize the role of the vector  $b$ ; thus  $A^i \langle 0 \rangle$  indicates  $A$  with its  $i$ th column zeroed out. Observe that  $A_i A_i^T = A^i \langle 0 \rangle A^T$ , by comparing corresponding entries.

Extend  $A$  to a nonsingular  $n \times n$  matrix  $\begin{pmatrix} A \\ C \end{pmatrix}$ , where  $C$  is an  $(n - m) \times n$  matrix whose rows form an orthonormal basis for the orthogonal complement of the row space of  $A$ . That is, each row of  $C$  has norm 1 and is orthogonal to all other rows of  $\begin{pmatrix} A \\ C \end{pmatrix}$ . We have

$$\begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} AA^T & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^i \langle b \rangle \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} A^i \langle b \rangle A^T & M \\ 0 & I \end{pmatrix},$$

where  $I$  is the  $(n - m) \times (n - m)$  identity matrix and  $M$  is some  $n \times (n - m)$  matrix. By substituting these computations into the definition of  $x_i$ , canceling the nonzero factor  $\det \begin{pmatrix} A \\ C \end{pmatrix}^T$ , and using the linearity of the determinant in its  $i$ th column, we obtain

$$x_i = \frac{\det \left( \begin{pmatrix} A^i \langle b \rangle \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T \right) - \det \left( \begin{pmatrix} A^i \langle 0 \rangle \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T \right)}{\det \left( \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^T \right)} = \frac{\det \begin{pmatrix} A^i \langle b \rangle \\ C \end{pmatrix} - \det \begin{pmatrix} A^i \langle 0 \rangle \\ C \end{pmatrix}}{\det \begin{pmatrix} A \\ C \end{pmatrix}} = \frac{\det \begin{pmatrix} A \\ C \end{pmatrix}^i \begin{pmatrix} b \\ 0 \end{pmatrix}}{\det \begin{pmatrix} A \\ C \end{pmatrix}},$$

By Cramer's rule,  $x$  is the solution to the linear system  $\begin{pmatrix} A \\ C \end{pmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix}$ , and hence  $x$  is a solution to  $Ax = b$ .

Solved also by J. Fuelberth & A. Gunawardena, J. H. Lindsey II, M. Sharma & P. G. Poonacha (India), WMC Problems Group, and the proposers.

### An Identity for Strongly Connected Digraphs

**10620** [1997, 870]. Proposed by James Propp, Massachusetts Institute of Technology, Cambridge, MA. A digraph on a vertex set  $V$  is a subset  $A \subseteq \{(v, w) : v, w \in V, v \neq w\}$  and is *strongly connected* if it is possible to get from any vertex  $a$  to every other vertex  $e$  by a finite succession of arcs  $(a, b)$ ,  $(b, c)$ ,  $\dots$ ,  $(d, e)$  in  $A$ . For  $n \geq 1$ , let  $E_n$  (respectively,  $O_n$ ) denote the number of strongly connected digraphs on the vertex set  $V = \{1, 2, \dots, n\}$  with an even (respectively odd) number of arcs. Show that  $E_n - O_n = (n - 1)!$  for all  $n \geq 1$ .

*Solution I by the proposer, currently at University of Wisconsin, Madison, WI.* The terminology of the problem statement is somewhat nonstandard. In common usage, a digraph is