10766. *Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let x, y, and z be nonnegative real numbers. Prove that

(a) $(x+y+z)^{x+y+z}x^x y^y z^z \le (x+y)^{x+y}(y+z)^{y+z}(z+x)^{z+x}$.

(b) $(x+y+z)^{(x+y+z)^2}x^{x^2}y^{y^2}z^{z^2} \ge (x+y)^{(x+y)^2}(y+z)^{(y+z)^2}(z+x)^{(z+x)^2}$.

SOLUTIONS

Cramer's Rule for Non-Square Matrices

10618 [1997, 768]. *Proposed by S. Lakshminarayanan, S. L. Shah, and K. Nandakumar, University of Alberta, Edmonton, Canada.* Let A be a real $m \times n$ matrix of full rank with $m < n$ and let b be a real $m \times 1$ matrix. For $1 \le i \le n$, define

$$x_i = \frac{\det(A_i^* A^T) - \det(A_i A_i^T)}{\det(AA^T)},$$

where A_i^* is obtained by replacing the ith column of A by b, and A_i is obtained by deleting the ith column of A. Show that $x = [x_1, \dots, x_n]^T$ is a solution to the linear system $Ax = b$.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We write $A^i\langle b\rangle$ instead of A_i^* to emphasize the role of the vector b; thus $A^i\langle 0\rangle$ indicates A with its ith column zeroed out. Observe that $A_i A_i^T = A^i\langle 0\rangle A^T$, by comparing corresponding entries.

Extend A to a nonsingular $n \times n$ matrix $\binom{A}{C}$, where C is an $(n-m)\times n$ matrix whose rows form an orthonormal basis for the orthogonal complement of the row space of A. That is, each row of C has norm 1 and is orthogonal to all other rows of $\binom{A}{C}$. We have

$$\begin{pmatrix} A \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} AA^T & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^i\langle b\rangle \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^T = \begin{pmatrix} A^i\langle b\rangle A^T & M \\ 0 & I \end{pmatrix},$$

where I is the $(n-m)\times(n-m)$ identity matrix and M is some $n \times (n-m)$ matrix. By substituting these computations into the definition of x_i, canceling the nonzero factor $\det\binom{A}{C}^T$, and using the linearity of the determinant in its ith column, we obtain

$$x_i = \frac{\det\left(\binom{A^i\langle b\rangle}{C}\binom{A}{C}^T\right) - \det\left(\binom{A^i\langle 0\rangle}{C}\binom{A}{C}^T\right)}{\det\left(\binom{A}{C}\binom{A}{C}^T\right)} = \frac{\det\binom{A^i\langle b\rangle}{C} - \det\binom{A^i\langle 0\rangle}{C}}{\det\binom{A}{C}} = \frac{\det\binom{A}{C}^i\left\langle\binom{b}{0}\right\rangle}{\det\binom{A}{C}},$$

By Cramer's rule, x is the solution to the linear system $\binom{A}{C}x = \binom{b}{0}$, and hence x is a solution to $Ax = b$.

Solved also by J. Fuelberth & A. Gunawardena, J. H. Lindsey II, M. Sharma & P. G. Poonacha (India), WMC Problems Group, and the proposers.

An Identity for Strongly Connected Digraphs

10620 [1997, 870]. *Proposed by James Propp, Massachusetts Institute of Technology, Cambridge, MA.* A *digraph* on a vertex set V is a subset $A \subseteq \{(v,w)\colon v, w \in V, v \ne w\}$ and is *strongly connected* if it is possible to get from any vertex a to every other vertex e by a finite succession of arcs $(a,b), (b,c), \dots, (d,e)$ in A. For $n \ge 1$, let E_n (respectively, O_n) denote the number of strongly connected digraphs on the vertex set $V = \{1, 2, \dots, n\}$ with an even (respectively odd) number of arcs. Show that $E_n - O_n = (n-1)!$ for all $n \ge 1$.

Solution I by the proposer, currently at University of Wisconsin, Madison, WI. The terminology of the problem statement is somewhat nonstandard. In common usage, a digraph is