to be included, of the number of ways to include it with even size minus the number of ways to include it with odd size. Let $b_m = E_m - O_m$; this is the contribution for a strong component of order m.

At this point we have

$$\sum(-1)^{e(D)} = \sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1!\ldots n_k!}\frac{1}{k!} b_{n_1}\ldots b_{n_k} \sum_{C} \prod_{i,j\in\binom{[k]}{2}} (E_{ij} - O_{ij}),$$

where the inner sum is over acyclic digraphs C on $[k]$, the notation $\binom{[k]}{2}$ stands for the set of all 2-element subsets of $[k]$, and E_{ij} and O_{ij}, respectively, denote the number of ways to have an even or odd number of edges between components i and j in the expansion of C.

When i and j are not adjacent in C, there is no edge in the expansion, and $E_{ij} - O_{ij} = 1$. When i and j are adjacent, there must be at least one edge in the expansion, and all such edges agree in direction with the edge in C. Eliminating the empty set yields $E_{ij} - O_{ij} = -1$. With the factor -1 for each edge of C, the product is $(-1)^{e(C)}$.

To further simplify the formula, we claim that $\sum_C(-1)^{e(C)} = (-1)^{k-1}$. We define an involution on the acyclic digraphs that pairs up digraphs with sizes differing by 1, and we show that the only unpaired digraph is the digraph C_k with arc set $\{(k, j)\colon 1 \le j \le k-1\}$. A *source* is a vertex with no incoming arc; a *predecessor* of j is a vertex i such that (i, j) is an arc.

Every acyclic digraph C has at least one source. Let i be the least source vertex. When $i \neq k$, we add or delete the arc (i, k); it remains true that i is the least source. In the remaining digraphs, the only source vertex is k. For these, let j be the highest vertex having a predecessor other than k. Add or delete the arc (k, j). It remains true that k is the only source and that j is the highest vertex with a predecessor other than k. The only digraph in which j does not exist is C_k (with $k-1$ edges), which completes the claim.

The coefficient of $x^n/n!$ in our generating function is now

$$\sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \binom{n}{\{n_i\}} \prod b_{n_i} \frac{(-1)^{k-1}}{k!}.$$

Given the exponential generating function $f(x) = b_n x^n/n!$, this yields $g(x) = 1 - e^{-f(x)}$. Since $g(x) = x$, we obtain $f(x) = -\ln(1-x)$, and hence $b_k = (k-1)!$.

Editorial comment. The proposer's proof is adapted from an analysis of the unsigned case due to V. A. Liskovec (On a recurrence method of counting graphs with labelled vertices, *Soviet Math. Dokl.* 10 (1969) 242–256) and explicated by E. M. Wright (The number of strong digraphs, *Bull. London Math. Soc.* 3 (1971) 348–350). Problem 6673 [1991, 965; 1994, 686] in this MONTHLY is the analogous problem for undirected graphs. The proposer notes the following consequence. If the sign of a disjoint union of strongly connected (simple) digraphs is $(-1)^{e+k}$, where e is the total number of arcs and k is the number of components, then the sum of the signs of all n-vertex disjoint unions of strongly connected digraphs is 0. He asks whether there is a simple direct proof of this corollary. One might also ask for a simple signed involution to prove the original claim directly.

Solved also by R. J. Chapman (U. K.).

Simultaneous Squares from Arithmetic Progressions

10622 [1997, 870]. *Proposed by M. N. Deshpande, Nagpur, India.* Find infinitely many triples (a, b, c) of positive integers such that a, b, c are in arithmetic progression and such that $ab+1$, $bc+1$, and $ca+1$ are perfect squares.

Solution I by Hansruedi Widmer, Nussbaumen, Switzerland. Let $a_0 = 1$, $a_1 = 4$, and $a_{n+2} = 4a_{n+1} - a_n$ for $n \ge 0$, and set $b_n = 2a_{n+1}$ and $c_n = a_{n+2}$. We claim that

(a_n, b_n, c_n) meets the conditions of the problem. By the recurrence, $c_n - b_n = b_n - a_n$, and we have arithmetic progressions.

To prove that $a_n c_n + 1$ is a perfect square, we prove by induction on n that $a_n a_{n+2} + 1 = a_{n+1}^2$. For $n = 0$, we have $1 \cdot 15 + 1 = 4^2$. For $n > 0$, we use the recurrence and the induction hypothesis to compute

$$a_n a_{n+2} + 1 = 4a_n a_{n+1} - a_n^2 + 1 = 4a_n a_{n+1} - a_{n-1}a_{n+1} = a_{n+1}^2.$$

Next, we use this formula and the recurrence to compute

$$\begin{aligned} a_n b_n + 1 &= 2a_n a_{n+1} + 1 = 2a_n a_{n+1} + a_{n+1}^2 - a_n a_{n+2} = a_{n+1}^2 - a_n(a_{n+2} - 2a_{n+1}) \\ &= a_{n+1}^2 - a_n(2a_{n+1} - a_n) = (a_{n+1} - a_n)^2. \end{aligned}$$

Similarly $b_n c_n + 1 = (a_{n+2} - a_{n+1})^2$, so all desired properties hold.

Solution II by Zachary Franco, Butler University, Indianapolis, IN. In $\mathbb{Z}[\sqrt{3}]$, the norm $\|r + s\sqrt{3}\| = r^2 - 3s^2$ is multiplicative and satisfies $\|2 + \sqrt{3}\| = 1$. Therefore, $\|(2 + \sqrt{3})^n\| = 1$. For the expansion $(2 + \sqrt{3})^n = r + s\sqrt{3}$, we thus have

$$3s^2 = r^2 - 1. \qquad (*)$$

For $n > 1$, the triple $(a, b, c) = (2s - r, 2s, 2s + r)$ is in arithmetic progression and satisfies $(2s - r)2s + 1 = (r - s)^2$, $(2s - r)(2s + r) + 1 = s^2$, and $2s(2s + r) + 1 = (r + s)^2$.

Editorial comment. The two solutions generate the same family of triples. Jan Kristian Haugland, John P. Robertson, and Ivan Vidav independently proved that this family contains all triples with $a \le b \le c$ that satisfy the conditions of the problem. Betsy Carper, Gary Hull, Lenny Jones, and Bonnie Wachhaus observed that for each triple in this family there is no fourth integer d such that a, b, c, d are in arithmetic progression and both $bd + 1$ and $cd + 1$ are perfect squares. Hence there is no quadruple of positive integers in arithmetic progression such that $ij + 1$ is a perfect square for all i, j in the quadruple.

David M. Bloom observed that this problem is related to problem 10238 [1992, 674; 1995, 275]. The published solution for that problem used the Pell equation $(*)$.

Solved also by I. Adler, M. Aissen, J. Anglesio (France), B. D. Beasley, J. C. Binz (Switzerland), D. M. Bloom, J. Bowring, J. T. Bruening, S. Byrd, R. J. Chapman (U. K.), J. Christopher, C. R. Diminnie, R. DiSario, H. Y. Far, J. K. Haugland (Norway), R. Heller, L. Jones et al., N. Komanda, J. Lee, N. F. Lindquist, C. Mack, L. G. Mans, A. S. Mittal, G. R. Mott, Y. Pan, M. Reekie, J. P. Robertson, H. Sedinger, N. C. Singer, W. R. Smythe, P. Trajovský (Czech Republic), I. Vidav (Slovenia), M. Vowe (Switzerland), X. Wang, C. H. Webster, P. Yiu, Anchorage Math Solutions Group, GCHQ Problems Group (U. K.), NSA Problems Group, NCCU Problems Group, SAS Maths Club (India), WMC Problems Group, and the proposer.

The Divisibility Poset Inside Itself

10623 [1997, 870]. *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.* Let $P = \{1, 2, 3, \ldots\}$, and let $|$ be the usual divisibility relation on P. For any $S \subseteq P$ and $n \in P$, let $S + n = \{s + n : s \in S\}$.

(**a**) Can one construct a subset S of P such that the poset $(S, |)$ is isomorphic to $(P, |)$, $(S + 1, |)$ is isomorphic to $(P, \le)$, and $(S + 2, |)$ is isomorphic to $(P, |)$?

(**b**) For which integers $n \ge 1$ can one find a subset T of P such that $(T, |)$, $(T + n, |)$, and $(P, |)$ are isomorphic posets?

Solution to (**a**) *by Nasha Komanda, Central Michigan University, Mt. Pleasant, MI.* The answer is yes. First we prove for positive integers x, y and for integers $a \ne \pm 1$ that

$$\gcd(a^x - 1, a^y - 1) = |a^{\gcd(x,y)} - 1|. \qquad (*)$$

We may assume that $x \ge y$. From $a^x - 1 = a^{x-y}(a^y - 1) + (a^{x-y} - 1)$ we obtain $\gcd(a^x - 1, a^y - 1) = \gcd(a^y - 1, a^{x-y} - 1)$. Also $\gcd(x, y) = \gcd(y, x - y)$. Thus $(*)$ follows by induction on $x + y$. If also x and y are odd, then

$$\gcd(a^x + 1, a^y + 1) = \gcd((-a)^x - 1, (-a)^y - 1) = |(-a)^{\gcd(x,y)} - 1| = |a^{\gcd(x,y)} + 1|.$$