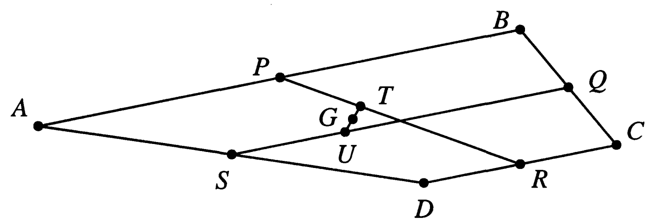
Quadrilateral Center of Gravity

10662 [1998, 464]. *Proposed by Joseph D. E. Konhauser and Stan Wagon, Macalester College, St. Paul, MN.* Find a construction for the center of gravity of the edges of a quadrilateral.

Solution by the Con Amore Problems Group, Royal Danish School of Educational Studies, Copenhagen, Denmark. If G is the center of gravity of the edges of the quadrilateral $ABCD$ then G is also the center of gravity of particles with masses proportional to the lengths of the edges AB, BC, CD, DA placed at the midpoints P, Q, R, S of these edges. Construct these midpoints. The center of gravity for the particles at P and R is the point T on PR such that $PT : TR = CD : AB$, and the center of gravity for the particles at Q and S is the point U on QS with $QU : US = DA : BC$, so we construct the points T and U. If T and U coincide, we have $G = T = U$. If not, then G is the center of gravity of particles with masses proportional to the lengths of the line segments $AB + CD$ and $BC + DA$ placed at T and U, respectively. The sum of two line segments is constructed by placing them end to end. Thus G is the point on TU with $TG : GU = (BC + DA) : (AB + CD)$, and we construct this point.



Solved also by M. Benedicty, M. Boase (U. K.), G. D. Chakerian, R. J. Chapman (U. K.), S. S. Kim (Korea), J. H. Lindsey II, A. Nijenhuis, V. Pambuccian, C. R. Pranesachar (India), A. Sasane (The Netherlands), J. Schaer (Canada), Anchorage Math Solutions Group, GCHQ Problems Group, and the proposers.

Logarithmic Convexity of Stirling's Ratio

10680 [1998, 666]. *Proposed by Harold G. Diamond, University of Illinois, Urbana, IL.* For $x > 0$ set $g(x) = x \log\left(\Gamma(x+1)/(x^x e^{-x}\sqrt{2\pi x})\right)$. Show that g is concave down on $(0, \infty)$.

Solution by Nathaniel Grossman, University of California, Los Angeles, CA. It is enough to show that $g''(x) < 0$ when $x > 0$. We begin with Binet's second expression for the gamma function, which we write in the form

$$\log\Gamma(x+1) = (x + \frac{1}{2})\log x - x + \frac{1}{2}\log(2\pi) + 2k(x), \qquad (*)$$

where $k(x) = \int_0^\infty \arctan(t/x)(e^{2\pi t} - 1)^{-1}\,dt$ (M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions,* Dover, New York, 1972, p. 258 (**6.1.50**)). From $(*)$ we find that $g(x) = 2xk(x)$, hence $g''(x) = 2\left(xk''(x) + 2k'(x)\right)$. Easily justified differentiation under the integral sign leads to

$$xk''(x) + 2k'(x) = -\int_0^\infty \frac{2t^3}{(x^2+t^2)^2(e^{2\pi t} - 1)}\,dt,$$

in which the right hand side is clearly negative.

Solved also by J. Anglesio (France), P. Bracken (Canada), D. Bradley, E. Camouzis (Greece), R. J. Chapman (U. K.), R. A. Groeneveld, D. Krüg, O. P. Lossers (The Netherlands), R. Martin (U. K.), A. McD. Mercer (Canada), P. Simeonov, A. Stadler (Switzerland), and NCCU Problems Group.