NOTES

Edited by **Jimmie D. Lawson and William Adkins**

On the Nelson Unit Distance Coloring Problem

Carsten Thomassen

In 1950 Nelson raised the problem of coloring the Euclidean plane in such a way that no two points of distance 1 receive the same color. How many colors are needed? This problem was often mentioned in Paul Erdős' famous lectures on unsolved combinatorial problems. The history of the problem is described in [**2**] and [**3**]. Clearly, three colors are needed. To see that four colors are needed, we consider seven points $x_1, x_2, \dots, x_7$ in the Euclidean plane such that the following pairs are of distance $1: x_1x_2, x_1x_3, x_1x_4, x_3x_4, x_2x_5, x_2x_6, x_5x_6, x_3x_7, x_4x_7, x_5x_7, x_6x_7$. It follows from the theorem of de Bruijn and Erdős [**1**] that the number of colors needed for the whole plane is the maximum number of colors needed for the finite subsets. The half-century old upper bound 7 is obtained by drawing an appropriate graph in the plane such that each face (region) is bounded by a cycle of (Euclidean) diameter less than one and then coloring each face and part of the boundary by the same color in such a way that only faces of distance > 1 receive the same color. We prove that colorings of this type always need at least 7 colors. More generally, 7 colors are needed for any surface and any metric of large diameter provided there are no short noncontractible curves and no short contractible curves whose interior have large area.

The upper bound 7 is obtained from a hexagonal tiling of the plane such that the hexagons are regular and of diameter slightly less than 1. All points inside a hexagon are colored with the same color. Two hexagons are colored differently if the distance between them is less than one. A coloring of this type will be called *nice*. More generally, we consider any metric space S, d such that S is a surface, i.e., S is an arcwise connected Hausdorff space such that each element of S has a neighborhood homeomorphic to an open disc in the Euclidean plane. We let G be a connected graph on S, i.e., the vertices of G are elements of S, and the edges of G are simple arcs on S that are pairwise disjoint except at a common vertex. Moreover, we assume that each face (i.e., arcwise connected component of $S \setminus G$) has diameter less than 1, is homeomorphic to a disc, and is bounded by a cycle in G. Now a *nice coloring* of S obtained from G is a coloring such that each color class is the union of faces (and part of their boundaries) such that the distance between any two of these faces is greater than 1. We define the *area* of subset A of S as the maximum number of pairwise disjoint open discs of radius $\frac{1}{2}$ that are contained in A. (If this maximum does not exist we say that A has infinite area.) We say that a simple closed curve C is *contractible* if $S \setminus C$ has precisely two arcwise connected components such that one of them is homeomorphic to an open disc in the Euclidean plane. That component is called the *interior* of C and is denoted *int*(C). (If S is a sphere, then *int*(C) denotes any component of $S \setminus C$ of smallest area). We prove that every nice coloring of S needs at least 7 colors provided there exists a natural number k such that (i), (ii), (iii) below hold.

(i) Every noncontractible simple closed curve has diameter at least 2.
(ii) If C is a simple closed curve of diameter less than 2, then the area of $int(C)$ is at most k.
(iii) The diameter of S is at least $12k + 30$.

If any of these conditions (i), (ii), (iii) is dropped, then the number of colors needed may decrease. Thus a thin two-way infinite cylinder has a nice 6-coloring, which shows that (i) cannot be omitted. Similarly, a thin one-way infinite cylinder (with a small disc pasted on the boundary of the cylinder to form the bottom) shows that (ii) cannot be omitted. Finally, (iii) cannot be omitted since any sphere of diameter less than 1 has a nice coloring in two colors.

D. R. Woodall **[5]** (see also **[4]** for a correction) has obtained a 6-color theorem related to the 7-color theorem in our Theorem 1.

2. A lemma on degrees in graphs. A *graph* G is a set $V(G)$ of elements called *vertices* and a set $E(G)$ of unordered pairs xy of vertices called *edges*. If the edge xy is present we say that *xy joins x* and y and that x and y are *neighbors*. The number of neighbors of x is the *degree* of x. A *path* from x to y is a graph consisting of distinct vertices $x_1, x_2, \dots, x_n$ and the edges $x_1x_2, x_2x_3, \dots, x_{n-1}x_n$ where $x_1 = x$, $x_n = y$. If we add the edge x_nx_1 we obtain a *cycle*. If x is a vertex, then $D_1(x)$ is the set of neighbors of x. More generally, if $n \geq 2$, then $D_n(x)$ is the set of vertices in $V(G) \setminus [\{x\} \cup D_1(x) \cup \cdots \cup D_{n-1}(x)]$ having a neighbor in $D_{n-1}(x)$. The subgraph of G induced by $\{x\} \cup D_1(x) \cup D_2(x) \cup \cdots$ is the connected component of G containing x. The graph G is *connected* if G has only one connected component. G is *locally finite* if $D_1(x)$ is finite for each vertex x in G. G is *locally connected* if for each vertex x, the subgraph of G induced by $D_1(x)$ is connected. G is *locally Hamiltonian* if, for each x in $V(G)$, G has a cycle with vertex set $D_1(x)$. The graph of the icosahedron is locally Hamiltonian and has 12 vertices all of degree 5. No larger connected graph has these properties.

Lemma 1. *If G is a connected, locally finite, and locally Hamiltonian graph with at least* 13 *vertices, then G has a vertex of degree at least* 6.

Proof: If no vertex has degree at least 6 we pick a vertex x of maximum degree. Clearly x has degree at least 3. If x has degree 3, then the subgraph of G induced by $\{x\} \cup D_1(x)$ is the graph of the tetrahedron, because G has a cycle with vertex set $D_1(x)$. Since G has maximum degree 3, there is no vertex in $D_2(x)$. Since G is connected, G is the graph of the tetrahedron, contrary to the assumption that G has at least 13 vertices. If x has degree 4, then we consider a cycle in $D_1(x)$ and conclude that each vertex y in $D_1(x)$ has at most one neighbor z in $D_2(x)$. Since G has a cycle with vertex set $D_1(y)$, z has at least three neighbors in $D_1(x)$. So, there are at most 4 edges from $D_1(x)$ to $D_2(x)$, and every vertex in $D_2(x)$ has at least three neighbors in $D_1(x)$. Hence $D_2(x)$ has at most one vertex z. Since G has a cycle with vertex set $D_1(z)$, it follows that $D_3(x) = \emptyset$. So, G has at most 6 vertices, a contradiction. We may therefore assume that x has degree 5.

Each vertex y in $D_1(x)$ has at most two neighbors in $D_2(x)$, because a cycle with vertex set $D_1(y)$ shows that y has at least two neighbors in $D_1(x)$, Since G has a cycle with vertex set $D_1(y)$, every neighbor z of y in $D_2(x)$ has at least two neighbors in $D_1(x)$. Now z cannot have two or more neighbors in $D_3(x)$ because then a cycle with vertex set $D_1(z)$ shows that z has at least two neighbors in

$D_2(x)$, that is, z has a total of at least 6 neighbors, a contradiction. So z has at most one neighbor in $D_3(x)$ and that neighbor has at least three neighbors in $D_2(x)$. Since there are at most 10 edges from $D_1(x)$ to $D_2(x)$, and every vertex in $D_2(x)$ has at least two neighbors in $D_1(x)$, it follows that $D_2(x)$ has at most 5 vertices. Hence there are at most 5 edges from $D_2(x)$ to $D_3(x)$. Since each vertex in $D_3(x)$ has at least three neighbors in $D_2(x)$, it follows that $D_3(x)$ has at most one vertex, and $D_4(x) = \emptyset$. Hence G has at most 12 vertices, a contradiction that completes the proof. ∎

3. A 7-color theorem

Theorem 1. *Let G be a connected graph on a surface S satisfying (i), (ii), and (iii). Then every nice coloring needs at least 7 colors.*

Proof: Suppose (reductio ad absurdum) that there exists a coloring using at most 6 colors. We define the *map graph* $M = M(G, S)$ as the graph whose vertices are the faces of G such that two vertices in M are neighbors if and only if the corresponding facial cycles in G intersect. Consider any vertex x of M and let C_x be the corresponding facial cycle in G. We choose an orientation of C_x and let $x_1, x_2, \dots, x_k, x_1$ be the vertices in $D_1(x)$ listed in the order that they are encountered when we traverse C_x. We now explain the idea behind the proof. We consider first the particularly nice case where, for each vertex x, all vertices $x_1, x_2, \dots, x_k$ are distinct. In that case, M is locally Hamiltonian. Since the surface S is arcwise connected, it follows that M is connected. Since S has diameter greater than 13, M has more than 12 vertices, and hence, by Lemma 1, M has a vertex of degree at least 6. Now x and its neighbors must have distinct colors because x corresponds to a face of diameter < 1 on S. This contradiction completes the proof in the particularly nice case where M is locally Hamiltonian.

However, a vertex may appear several times in the sequence $x_1, x_2, \dots, x_k$ above, and some more careful analysis is needed. We omit those appearances (except possibly one) of x_i for which C_{x_i} and C_x have only a vertex in common. In other words, if x_i appears more than once in the new sequence, then we list only those appearances such that C_{x_i} and C_x share an edge. Then any two consecutive vertices in the sequence $x_1, x_2, \dots, x_k, x_1$ are neighbors in M and so M is locally connected. It follows that $M - x$ (that is, M with x and the edges incident with x removed) is connected. Moreover, if y is any other vertex of M, then $M - x - y$ is connected unless y appears twice in the sequence $x_1, x_2, \dots, x_k$, that is, C_x and C_y have at least two edges in common.

Consider now two vertices x and y such that C_x and C_y have at least two edges e and f in common (that is, $y = x_i = x_j$ for $1 \le i < j - 1 < k - 1$). Let R be a simple closed curve in the faces bounded by C_x and C_y such that R crosses each of e and f precisely once and has no other point in common with G. By (i), R is contractible. Hence $M - x - y$ is disconnected. We say that$\{x, y\}$ is a *2-separator* in M. For each vertex z in M such that C_z is in $int(R)$ and has color 1, we pick a point P_z in $int(C_z)$, By (ii), there are at most k points P_z and hence there are altogether at most $6k$ vertices z such that $int(C_z) \subseteq int(R)$. We define $int(M, x, y)$ as the subgraph of $M - x - y$ induced by all those vertices z in M such that C_z is in $int(R)$ for some R. Then each connected component of $int(M, x, y)$ has at most $6k$ vertices. Since S has diameter at least $12k + 3$ it follows that G has two vertices whose graph distance is at least $12k + 2$. Hence $M - x - y$ has some component that is not in $int(M, x, y)$. We claim that $M - x - y$ has precisely one such component, which we call $ext(M, x, y)$. To see this, let $e_1, e_2, \dots, e_m$ be the edges in $C_x \cap C_y$ occuring in that cyclic order on C_x. Then $e_1, \ldots, e_m$ divide $D_1(x) \setminus \{y\}$ into m classes $A_1, A_2, \ldots, A_m$. By letting $\{e, f\} = \{e_i, e_{i+1}\}(1 \le i \le m)$ in the preceding argument, we conclude that for each $i = 1, 2, \ldots, m$, either $A_i \subseteq int(M, x, y)$ or $A_i \cap int(M, x, y) = \emptyset$. Since the former cannot hold for each $i \in \{1, 2, \ldots, m\}$, the latter must hold for some i, and hence the former holds for all other i in $\{1, 2, \ldots, m\}$. Summarizing, for any two vertices x, y in M, $M - x - y$ has precisely one connected component $ext(M, x, y)$ with more than $6k$ vertices.

If $\{u, v\}$ is a 2-separator in M such that either x or y or both is in $int(M, u, v)$, then clearly $int(M, x, y) \subset int(M, u, v)$. (To see this, we use the properties of M established previously and forget about S.) If no such 2-separator $\{u, v\}$ exists, then we say that $\{x, y\}$ is a *maximal 2-separator* and that xy is a *crucial edge*. Since each connected component of $int(M, x, y)$ has at most $6k$ vertices, then a maximal 2-separator exists (provided a 2-separator exists). Let H be the subgraph of M obtained by deleting $int(M, x, y)$ for each maximal 2-separator $\{x, y\}$. Then $H \neq \emptyset$. Moreover, H is connected since a shortest path in M between two vertices in H never uses vertices in $int(M, x, y)$. Similarly, H is locally connected. We claim that H is locally Hamiltonian. Consider again a vertex x in H and the sequence $x_1, x_2, \ldots, x_k, x_1$ in $D_1(x)$ (taken in M). If this sequence forms a Hamiltonian cycle in $D_1(x)$ in H, we have finished. By the definition of H, $k \ge 3$. So assume that $x_i = x_j$ where $1 \le i < j - 1 < k - 1$. Then $\{x, x_i\}$ is a 2-separator and the notation can be chosen such that $int(M, x, x_i\}$ contains all the vertices $x_{i+1}, x_{i+2}, \ldots, x_{j-1}$. We repeat this argument for each pair i, j such that $x_i = x_j$ where $1 \le i < j - 1 < k - 1$. Then the vertices in $x_1, x_2, \ldots, x_k, x_1$ that remain after we delete all vertices in the interiors of the 2-separators form a cyclic sequence with no repetitions. As H is connected and locally connected and has at least three vertices (by (iii)), the preceding reduced cyclic sequence has at least two distinct vertices. It cannot have precisely two vertices u, v because then $H - u - v$ is disconnected, and hence $M - u - v$ is disconnected (because M is obtained from H by "pasting graphs on edges of H"). Since one of the edges xu or xv is crucial (because $D_1(x)$ is smaller in H than in M), the maximality property of the 2-separator $\{x, u\}$ or $\{x, v\}$ implies that $ext(M, u, v)$ is the connected component of $M - u - v$ containing x. For each vertex z in that component, M has a path of length at most $6k$ from z to either x, u, or v. Hence M has diameter at most $12k + 1$, a contradiction that proves that H is locally Hamiltonian.

If H has a vertex x of degree at least 6 we have finished because x and its neighbors must have different colors in the nice coloring. So, we may assume that each vertex of H has degree at most 5. By Lemma 1, H has at most 12 vertices. Hence H has at most 30 edges. Since M is obtained from H by "pasting" $int(M, x, y\}$ on the crucial edge xy for each crucial edge of H, we conclude that the diameter of M is at most $12k + 29$, a contradiction to (iii).

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Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark
C.Thomassen@mat.dtu.dk