simplest impossible case is the sequence $+, 0, -, 0, +$, corresponding to the polynomial $ax^4 - bx^2 + c = 0$, which has $-x$ as a root if it has x as a root and thus cannot simultaneously have no positive and two negative roots.

Even if no signs are zero, it may not be possible to obtain simultaneously all admissible numbers of positive and negative roots. For example, the sequence $+, -, -, -, +$ has two positive and two negative sign changes. It is possible for a polynomial with this sign sequence to have two negative or zero positive real roots, but not both simultaneously. A fourth-degree polynomial with only two negative real roots for which the sum of the roots was positive could be factored as $a(x^2 + bx + c)(x^2 - sx + t)$ with $a, b, c, s, t > 0$, $s^2 < 4t$, and $b^2 \geq 4c$. The product of these factors is $a(x^4 + (b - s)x^3 + (t + c - bs)x^2 + (bt - cs)x + st)$. To get the correct sign sequence, we need $b < s$ and $bt < cs$, which gives $b^2 t < s^2 c$ and thus $b^2/c < s^2/t$. But we have $b^2/c \geq 4 > s^2/t$.

This counterexample provides a negative answer to the question raised in **[1]** whether it is possible to get a polynomial with an arbitrary sign sequence and any simultaneous numbers of positive and negative roots allowed by Descartes' Rule of Signs. This suggests a new conjecture: the only possible numbers of positive and negative roots are the maximum values permitted by Descartes' Rule of Signs in a sequence obtained by changing some of the internal signs to zeros as in Theorem 2. The above cases and the analogous $+, +, -, +, +$ confirm the conjecture for degree 4.

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Integrating Polynomials in Secant and Tangent

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This note provides a relatively painless way to integrate arbitrary polynomials in secant and tangent without invoking integration by parts or anything beyond elementary polynomial and trigonometric identities. The techniques involved also introduce some of the ideas behind the construction of Laurent polynomials, although the manner in which they do so is rather indirect. We begin with a theorem that covers almost all possibilities.

Theorem 1. *For each polynomial $P(s, t)$ in two variables, there are polynomials F and G in one variable and a constant c such that*

$$\int P(\sec x, \tan x) \sec x \, dx = F(u) - G(v) + c \ln(u) + C$$

where $u = \sec x + \tan x$ and $v = \sec x - \tan x$.

Proof: Once we define $u = \sec x + \tan x$ and $v = \sec x - \tan x$, it is easy to check that $\sec x = (u+v)/2$, $\tan x = (u-v)/2$, and $uv = \sec^2 x - \tan^2 x = 1$. (The traditional construction of Laurent polynomials involves quotienting the two-variable polynomial ring by the two-sided principal ideal generated by the polynomial $xy - 1$. Since $uv = 1$, the polynomials in secant and tangent can be viewed as Laurent polynomials in the single variable $u = \sec x + \tan x$. This is an abstract algebra explanation for the simplifications that follow.) If we replace $\sec x$ and $\tan x$ with their equivalents in terms of u and v, then the polynomial P in $\sec x$ and $\tan x$ becomes a polynomial in u and v instead. Moreover, since $uv = 1$, we can replace any instance of uv by 1, quickly reducing any monomial containing both variables to one that contains at most a single variable. In other words, the resulting polynomial in u and v can always be written in the form $f(u)u + g(v)v + c$ where f and g are polynomials of a single variable and c is a constant.

Next consider the differentials du and dv. Since $du = \sec x \tan x + \sec^2 x\, dx$ and $dv = \sec x \tan x - \sec^2 x\, dx$, we find that $\sec x\, dx = (1/u)\, du = -(1/v)\, dv$. We are now ready to calculate the original integral.

$$\begin{aligned}\int P(\sec x, \tan x)\sec x\, dx &= \int (f(u)u + g(v)v + c)\sec x\, dx \\ &= \int f(u)\, du - \int g(v)\, dv + c\int \frac{1}{u}\, du \\ &= F(u) - G(v) + c\ln|u| + C\end{aligned}$$

where $F(u)$ and $G(v)$ represent the antiderivatives of the polynomials $f(u)$ and $g(v)$, respectively. ■

Example 2. Consider the integral $\int 16\sec^5 x\, dx$. We find that

$$\begin{aligned}P(s,t) &= 16s^4 \\ P\left(\frac{u+v}{2}, \frac{u-v}{2}\right) &= (u+v)^4 \\ &= u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 \\ &= u^4 + 4u^2 + 6 + 4v^2 + v^4\end{aligned}$$

Thus $f(u) = u^3 + 4u$, $g(v) = 4v + v^3$, and $c = 6$; we immediately conclude that

$$\int 16\sec^5 x\, dx = \left(\frac{u^4}{4} + 2u^2\right) - \left(2v^2 + \frac{v^4}{4}\right) + 6\ln|u| + C$$

where $u = \sec x + \tan x$ and $v = \sec x - \tan x$. The standard approach would involve performing integration by parts twice in order to reduce the exponent of the integrand.

Theorem 1 is nearly comprehensive in the sense that the only monomials in secant and tangent that are not covered are the constant term and those of the form $\tan^n x$. Closed forms for these integrals exist [**1**], but a little bit of trigonometry brings them within the reach of Theorem 1. Consider, for example, the monomial $\tan^7 x$. By repeatedly applying the identity $\tan^2 x = \sec^2 x - 1$ we see that

$$\begin{aligned}\tan^7 x &= \tan^5 x \sec^2 x - \tan^5 x \\ &= \tan^5 x \sec^2 x - \tan^3 x \sec^2 x + \tan^3 x \\ &= \tan^5 x \sec^2 x - \tan^3 x \sec^2 x + \tan x \sec^2 x - \tan x\end{aligned}$$

In the final expression, the first three terms can be integrated using Theorem 1, so that only $\tan x$ remains. More generally, given any polynomial in secant and tangent the pure powers of tangent can be modified in this way so that the result is the sum of a constant term, a constant multiple of $\tan x$, and a polynomial to which Theorem 1 can be applied. This completes the proof of the following result.

Theorem 3. *For each polynomial $P(s, t)$ in two variables, there are polynomials F and G in one variable and constants a, b, and c such that*

$$\int P(\sec x, \tan x)\, dx = F(u) - G(v) + a \ln|u| - b \ln|\cos x| + cx + C$$

where $u = \sec x + \tan x$ and $v = \sec x - \tan x$.

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