The Divisible Differences Property: 10553

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10709. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* Let X be a standard normal random variable, and choose $y > 0$. Show that

$$e^{-ay} < \frac{Pr(a \le X \le a+y)}{Pr(a-y \le X \le a)} < e^{-ay+(1/2)ay^3}$$

when $a > 0$. Show that the reversed inequalities hold when $a < 0$.

10710. *Proposed by Bogdan Suceava, Michigan State University, East Lansing, MI.* Let ABC be an acute triangle with incenter I, and let D, E, and F be the points where the circle inscribed in ABC touches BC, CA, and AB, respectively. Let M be the intersection of the line through A parallel to BC and DE, and let N be the intersection of the line through A parallel to BC and DF. Let P and Q be the midpoints of DM and DN, respectively. Prove that A, E, F, I, P, and Q are on the same circle.

SOLUTIONS

When O-H-I Is Isosceles

10547 [1996, 695]. *Proposed by Dan Sachelarie, ICCE Bucharest, and Vlad Sachelarie, University of Bucharest, Bucharest, Romania.* In the triangle ABC, let O be the circumcenter, H the orthocenter, and I the incenter. Prove that the triangle OHI is isosceles if and only if

$$\frac{a^3+b^3+c^3}{3abc} = \frac{R}{2r}.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria. We denote by MPV the reference D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities,* Kluwer, 1989. Neither IO nor HI is ever as large as HO [MPV, p. 288], so the only way triangle IHO can be isosceles is if $IO = HI$. Also $IO^2 = R^2 - 2Rr$ [MPV, p. 279] and $HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$ [MPV, p. 280], where s is the semiperimeter. Hence $HI = IO$ if and only if $R^2 - 2Rr = 4R^2 + 4Rr + 3r^2 - s^2$. This rearranges to $2s(s^2 - 3r^2 - 6Rr)/12Rrs = R/2r$, or, using $abc = 4Rrs$ [MPV, p. 52] and $a^3+b^3+c^3 = 2s(s^2 - 3r^2 - 4Rr)$ [MPV, p. 52], to $(a^3+b^3+c^3)/3abc = R/2r$.

Editorial comment. Another condition equivalent to $HI = IO$, given in problem E2282 [1971, 196; 1972, 397] from this MONTHLY, is that ABC has one angle equal to 60°.

Solved also by J. Anglesio (France), R. Barbara (Lebanon), F. Bellot Rosado (Spain), C. W. Dodge, J. S. Frame, Z. Franco, M. S. Klamkin (Canada), W. W. Meyer, V. Mihai (Canada), C. R. Pranesachar (India), B. Prielipp, V. Schindler (Germany), I. Sofair, M. Tabaâ (Morocco), T. V. Trif (Romania), M. Vowe (Switzerland), GCHQ Problems Group (U. K.), and the proposers.

The Divisible Differences Property

10553 [1996, 809]. *Proposed by Bjorn Poonen, Mathematical Sciences Research Institute, Berkeley, CA, Jim Propp, Massachusetts Institute of Technology, Cambridge, MA, and Richard Stong, Rice University, Houston, TX.* Say that a sequence $\langle q \rangle = q_1, q_1, q_2, \ldots$ of integers has the *divisible differences property* if $(n-m)|(q_n - q_m)$ for all n and m.
(a) Show that if $\langle q \rangle$ has the divisible differences property and $\limsup |q_n|^{1/n} < e - 1$, then there is a polynomial Q such that $q_n = Q(n)$.
(b) Show that there is a sequence $\langle q \rangle$ that has the divisible differences property and satisfies $\limsup |q_n|^{1/n} \le e$, for which q_n is not given by a polynomial in n.
(c)* Is it true that $\limsup |q_n|^{1/n} \ge e$ for all non-polynomial $\langle q \rangle$ with the divisible differences property?

Solution of parts (**a**) *and* (**b**) *by the GCHQ Problem Solving Group, Cheltenham, U. K.* Let $L_n = \text{lcm}\{1, \ldots, n\}$. We need three facts.

Fact 1: $\log(L_n) \sim n$ as $n \to \infty$.

Proof: If p is prime and $p \le n$, then $p^{\lfloor \log_p(n) \rfloor}$ is the highest power of p that divides n. Therefore, $L_n = \prod_{p \le n} p^{\lfloor \log_p(n) \rfloor} < \prod_{p \le n} p^{\log_p(n)} = \prod_{p \le n} n = n^{\pi(n)}$. Thus, $\log(L_n) < \pi(n) \log n \sim n$, the latter following from the Prime Number Theorem. Conversely, if $1/2 < r < 1$, then $L_n > \prod_{n^r < p \le n} p > \prod_{n^r < p \le n} n^r = n^{r(\pi(n) - \pi(n^r))}$. Taking logarithms and using the Prime Number Theorem yields $\log(L_n) > r(\pi(n) - \pi(n^r)) \log n \sim r\left(\frac{n}{\log n} - \frac{n^r}{r \log n}\right) \log n = rn - n^r \sim rn$. Since r is arbitrary, $\log(L_n) \sim n$. $\square$

Fact 2: Given $q_0, q_1, \ldots, q_{n-1}$, the choices for q_n such that $(n-m) | (q_n - q_m)$ for $0 \le m < n$ lie in the same congruence class modulo L_n.

This is a consequence of the Chinese Remainder Theorem.

Fact 3: If $q_0, q_1, \ldots, q_{n-1}$ has the divisible differences property, and if r_n is the value found by fitting a minimum degree polynomial to $q_0, q_1, \ldots, q_{n-1}$ and extrapolating, then $r_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} q_{n-i}$ and $(n - m) | (r_n - q_m)$ for $0 \le m < n$.

Proof: The formula for r_n is just a restatement of the property that the nth difference of $q_0, q_1, \ldots, q_{n-1}$ is 0. We prove the rest by induction on n, it being trivial when $n = 1$.

The first difference of $q_0, q_1, \ldots, q_{n-1}$ is a sequence of length $n - 1$ with the divisible differences property, and its polynomial extrapolation is $r_n - q_{n-1}$. By the inductive hypothesis, $m | (r_n - q_{n-1}) - (q_{n-m} - q_{n-m-1})$ for $m > 0$, and by the hypothesis on $q_0, q_1, \ldots, q_{n-1}$ we have $m | (q_{n-1} - q_{n-m-1})$. Hence $m | (r_n - q_{n-m})$, so we may assume $m = 0$. By subtracting q_0 from the rest of the sequence, we may assume $q_0 = 0$, so

$$r_n = \sum_{i=1}^{n-1} (-1)^{i+1} \binom{n}{i} q_{n-i}. \tag{1}$$

Now $n | (n - i)\binom{n}{i}$ since $(n - i)\binom{n}{i} = n\binom{n-1}{i}$, and $(n - i) | q_{n-i}$, so n divides each term of the sum in (1) and therefore r_n. $\square$

We can restate the divisible differences property as: $L_n | (q_n - r_n)$ for all n.

(**a**) If $\limsup |q_n|^{1/n} < e - 1$, then there is an $\epsilon > 0$ and an integer n_0 such that $|q_n|^{1/n} < e - 1 - 3\epsilon$ for $n > n_0$. From this and Fact 3, there is an $n_1 > n_0$ such that $|r_n| < \sum_{i=n_0}^{n} \binom{n}{i} (e - 1 - 2\epsilon)^{n-i} < (e - 2\epsilon)^n < (e - \epsilon)^n / 2$ for $n > n_1$. Using Fact 1, there is an $n_2 > n_1$ such that $n > n_2$ implies that $\log(L_n) > n \log(e - \epsilon)$ (that is, $L_n > (e - \epsilon)^n$). If $n > n_2$ and $q_0, q_1, \ldots, q_{n-1}$ has the divisible differences property, r_n is thus the only value for q_n that would extend the property and also have absolute value less than $(e - \epsilon)^n/2$. Since $|q_n| < (e - 1 - 3\epsilon)^n$, it follows that $q_n = r_n$.

This is the inductive step in a proof that the minimum degree polynomial that fits $q_0, \ldots, q_{n_2}$ fits q_n for all n.

(**b**) Set $q_0 = 0$. For each positive n, set q_n to be congruent to r_n mod L_n but of opposite sign and having magnitude less than L_n. If $r_n = 0$, however, set $q_n = L_n$. Since q_n never equals r_n, q_n is not given by a polynomial. Since $|q_n| \le |L_n|$, we have $\limsup |q_n|^{1/n} \le e$.

Editorial comment. The results of parts (**a**) and (**b**) appear in I. Ruzsa, On congruence preserving functions (in Hungarian), *Mat. Lapok* 22 (1971) 125-134 (*Math. Reviews*, vol. 48, #2044). Part (**a**) generalizes problem 4 from the 1995 USA Mathematical Olympiad, which imposed the stronger condition that q_n is bounded by a polynomial. No solutions were submitted for part (**c**).

Parts (**a**) and (**b**) solved also by R. J. Chapman (U. K.), K. S. Kedlaya, and the proposers.