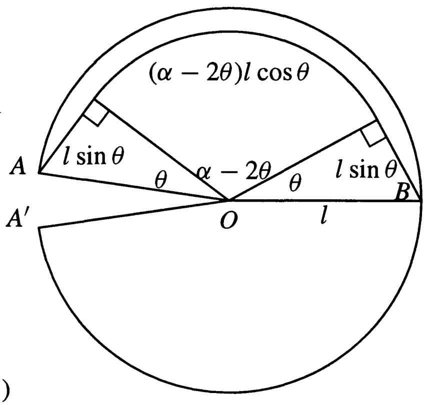
Mathematics for Hikers

10557 [1996, 902]. *Proposed by Nick MacKinnon, Winchester College, Winchester, U. K.* Naismith's rule allows walkers to compute the time for their journeys. The time is given by allowing a walking speed of 4 km/hr, but adding an extra minute for each 10m of ascent. A conical mountain has base radius 1650m and vertical height 520m. Points A and B are diametrically opposite at the base of the mountain. How should a path be constructed between between A and B on the surface of the mountain that minimizes the time taken to walk from A to B?

Solution by the proposer. The surface of the mountain can be unrolled isometrically onto a plane, forming a sector (of angle $2\alpha = 165/173 \cdot 2\pi$ radians) of a circle (of radius $l = 1730$m, the slant height of the mountain), as shown in the figure at right.



Call the time-minimising path the *Naismith geodesic* for the cone. This geodesic must reach some maximum height h. The figure shows a potential Naismith geodesic with maximum height $h = 520(1 - \cos\theta)$ meters. It is composed of a circular arc following the contour with height h, together with tangents joining the arc to A and B. No alternative path taking less time attains the maximum height h of the given path since such a path must at least touch the circular arc, must not cross the circular arc, and must leave the circular arc without subsequent reascent. The Naismith geodesic must therefore be a path of the given shape. The length of such a path is $2l\sin\theta + (\alpha - 2\theta)l\cos\theta$, and Naismith's rule gives a time of

$$t(\theta) = 0.015(2l\sin\theta + (\alpha - 2\theta)l\cos\theta) + 520(1 - \cos\theta)/10$$

minutes for the journey. The lone critical value of $t(\theta)$ for $0 < \theta < \alpha/2$ occurs when $t'(\theta) = (52 - 0.015(\alpha - 2\theta)l)\sin\theta = 0$ at $\theta^* = (\alpha - 52/(0.015l))/2 \doteq .49623$ radians. This gives the optimal time $t(\theta^*) = 76.71037$ minutes. A path around the bottom of the cone takes $t(0) = 77.75442$ minutes, while the path of shortest distance takes $t(\alpha/2) = 99.98929$ minutes.

Solved also by J. Anglesio (France), J. E. Dawson (Australia), P. G. Kirmser, J. H. Lindsey II, R. Reynolds & M. Martinez, and P. Straffin.

A Matter of Adjustment

10558 [1996, 902]. *Proposed by Zhang Chengyu, Hubei University, Wuhan, China.* Let p be a prime, and let k be a positive integer. Let $a_1, a_2, \ldots, a_{p^k}$ be any p^k integers. We define the *adjustment* of these integers to be the p^k integers $b_1, b_2, \ldots, b_{p^k}$, where $b_j = a_{j+1} + a_{j+2} + \cdots + a_{j+p}$, interpreting subscripts modulo p^k. For example, if $p = 2$ and $k = 2$, one adjustment of 1, 1, 3, 4 gives 4, 7, 5, 2. Prove that after p^k adjustments of $a_1, a_2, \ldots, a_{p^k}$, the list consists entirely of integers divisible by p.

Solution I by Thomas Jager, Calvin College, Grand Rapids, MI. We prove a stronger statement. Given an integer vector $v = (v_1, \ldots, v_{p^k})$, define the v-adjustment of $(a_1, \ldots, a_{p^k})^T$ to be $(b_1, \ldots, b_{p^k})^T$, where $b_j = v_1 a_{j+1} + v_2 a_{j+2} + \cdots + v_{p^k} a_{j+p^k}$, again treating subscripts modulo p^k. As a transformation, the v-adjustment is represented by the matrix $A = v_1 S + v_2 S^2 + \cdots + v_{p^k} S^{p^k}$, where S is the permutation matrix for a cyclic shift by