one position. Hence

$$A^{p^k} = \left(v_1 S + \cdots + v_{p^k} S^{p^k}\right)^{p^k} \equiv v_1^{p^k} I + \cdots + v_{p^k}^{p^k} I \equiv (v_1 + \cdots + v_{p^k})^{p^k} I \pmod p.$$

Thus if $v_1 + \cdots + v_{p^k} \equiv 0$ modulo p, then p^k applications of the v-adjustment matrix produces a vector of integers divisible by p. In the problem statement, the vector v consists of p ones and $p^k - p$ zeros.

Solution II by J. H. van Lint, Eindhoven University of Technology, Eindhoven, the Netherlands. Starting with a_0, we construct an infinite sequence with $a_i = a_{i+p^k}$. Over the field $\mathbb{F}_p$, we consider the formal power series $f(x) = \sum_{i=0}^{\infty} a_i x^i = A(x)/(1 - x^{p^k})$, where $A(x) = \sum_{i=0}^{p^k-1} a_i x^i$ is a polynomial of degree less than p^k.

After one adjustment, the terms $b_0, b_1, \ldots$ are the coefficients of $x^{p+1}, x^{p+2}, \ldots$ in the formal power series for

$$(x + x^2 + \cdots + x^p) f(x) = \frac{x(1 - x^p)}{1 - x} f(x) = x(1 - x)^{p-1} f(x).$$

The result of n adjustments is the list of coefficients of $x^{n(p+1)}, x^{n(p+1)+1}, \ldots$ in the formal power series for

$$x^n (1 - x)^{n(p-1)} f(x) = \frac{x^n (1 - x)^{n(p-1)}}{1 - x^{p^k}} A(x),$$

which is a polynomial of degree less than np if $n(p-1) \geq p^k$. Thus the list consists entirely of integers divisible by p after n adjustments if $n \geq p^k/(p-1)$. Noting that

$$\frac{p^k - 1}{p - 1} + 1 = \frac{p^k}{p - 1} + \frac{p - 2}{p - 1}$$

is the least integer greater than or equal to $p^k/(p-1)$, we see that the list consists entirely of integers divisible by p after n adjustments if $n \geq (p^k - 1)/(p - 1) + 1$. As this is at most p^k, the desired result follows.

Editorial comment. David Callan proved that for positive m the list consists entirely of integers divisible by p^{m-1} after mp^{k-1} adjustments. In particular, after p^k adjustments the list consists entirely of integers divisible by p^{p-1}. Another consequence is that the list consists entirely of integers divisible by p after $2p^{k-1}$ adjustments, but this is not as strong as the result proved by van Lint.

Solved also by D. Beckwith, A. E. Caicedo Núñez (Colombia), D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), W. Janous (Austria), K. S. Kedlaya, J. H. Lindsey II, R. Martin (Germany), A. Nijenhuis, J. C. Smith, H.-T. Wee (Singapore), GCHQ Problems Group (U. K.), and the proposer.

Sets with Fixed Nim-Sum

10564 [1997, 68]. *Proposed by Proposed by Aviezri Fraenkel, Weizmann Institute of Science, Rehovot, Israel.* The *Nim-sum* of two positive integers with binary expansions $\sum_{i \geq 0} a_i 2^i$ and $\sum_{i \geq 0} b_i 2^i$ is the number with binary expansion $\sum_{i \geq 0} c_i 2^i$, where a_i, b_i, c_i are in $\{0, 1\}$ and $c_i \equiv a_i + b_i \bmod 2$. Let n be a positive integer, and let j be a nonnegative integer. How many of the 2^n subsets of $\{1, 2, \ldots, n\}$ have the property that their elements have Nim-sum equal to j?

Solution by Reiner Martin, Deutsche Bank, London, U. K. Let $[n] = \{1, 2, \ldots, n\}$, and let Δ denote the symmetric difference operation. Let $k = \lceil \log_2(n + 1) \rceil$. There exists a subset of $[n]$ whose elements have Nim-sum j only if $0 \leq j < 2^k$. We claim that the number of such subsets does not depend upon j and thus that this number is 2^{n-k} for each such j.

To prove this, let $\sum_{i\geq 0} a_i 2^i$ be the binary expansion of j, and let $A_j = \{2^j\colon\ a_j = 1\}$. For each $A \subseteq [n]$ with Nim-sum 0, let $f(A) = A\Delta A_j$. Note that $f(A)$ has Nim-sum j. Since $(A\Delta A_j)\Delta A_j = A$, this map is a bijection into the set of subsets of $[n]$ with Nim-sum j.

Solved also by D. Beckwith, M. Benedicty, D. Berstein, J. C. Binz (Switzerland), M. Bowron, D. Callan, R. J. Chapman (U. K.), D. Donini (Italy), G. Gordon, R. Holzsager, K. S. Kedlaya, N. Komada, J. H. Lindsey II, J. Lorch, O. P. Lossers (The Netherlands), D. K. Nester, A. Nijenhuis, K. O'Bryant, M.-K. Siu (Hong Kong), J. H. Steelman, W. Stromquist, I. Vardi (Canada), H.-T. Wee (Singapore), M. Wolterman, Anchorage Math Solutions Group, GCHQ Problems Group (U. K.), NCCU Problems Group, NSA Problems Group, and the proposer.

Generalized Line Bingo

10565 [1997, 68]. *Proposed by D. M. Bloom, Brooklyn College, Brooklyn, NY, and Kenneth Suman, Winona State University, Winona, MN.* A rectangle is composed of mn squares arranged in m rows and n columns. In a certain game, the squares are selected one by one at random (without replacement). What is the expected number of selections until j columns of the rectangle are composed entirely of selected squares? (When $j = 1$, $m = 5$, and $n = 15$, this is the expected length of a type of bingo game known as a line game.)

Composite solution by the GCHQ Problems Group, Cheltenham, U. K. and the editors. For fixed m and n, the required expectation E_j equals $mn \prod_{i=j}^{n-1} mi/(mi+1)$.

For each instance of the game, we can continue selecting squares at random until all squares are selected. Thus it suffices to compute, over all permutations of the mn squares, the expected length of the initial segment that completes j columns. We compute for each square x the probability that it belongs to that initial segment. This is independent of x, so the expectation is mn times this probability.

Let A_i be the event that x belongs to the initial segment in which i columns are completed; note that $Pr(A_n) = 1$. The probability $Pr(A_j)$ is the product over $i \geq j$ of $Pr(A_i|A_{i+1})$.

We partition A_{i+1} into subevents that fix the trailing segment after the position where the $(i+1)$st column is completed. In such a subevent S, the identities of the first $i+1$ finished columns are fixed, but not which of these is last.

For permutations in S, let B be the set of squares consisting of the first i finished columns and the last square that completes the $(i+1)$st completed column. When $x \in B$, it is equally likely to occupy any of the $mi+1$ positions occupied by B, so the fraction of such permutations that belong to A_i is $mi/(mi+1)$.

When $x \notin B$, we can group the permutations by each fixed permutation of B. Now x is equally likely to fall into each of the $mi+1$ segments between members of B (or before the first). Again the fraction of these permutations that belong to A_i is $mi/(mi+1)$.

Editorial comment. The rows are unimportant. Víctor Hernández used linearity of expectation and the inclusion-exclusion principle to obtain a formula in the more general situation where the columns are sets of arbitrary size.

Solved also by R. J. Chapman (U. K.), D. A. Darling, V. Hernández (Spain), R. Holzsager, J. H. Lindsey II, P. W. Lindstrom, N. C. Singer, J. C. Smith, J. H. Steelman, Anchorage Math Solutions Group, and the proposer.

Ordered Trees and Stirling Numbers

10570 [1997, 69]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* An ordered tree is a rooted tree in which the children of each node form a sequence rather than a set. The height of an ordered tree is the number of edges on a path of maximum length starting at the root. Let $a(n,k)$ denote the number of ordered trees with n edges and height k, and let $S(n,k)$ be the Stirling number of the second kind (the number of partitions of $\{1, 2, \ldots, n\}$ into k nonempty parts). Note that $a(n,1) = S(n,1)$, since both numbers are 1. Show that (**a**) $a(n,2) = S(n,2)$, (**b**) $a(n,3) + a(n,4) = S(n,3)$, and (**c**)* generalize these observations.