To prove this, let $\sum_{i\geq 0} a_i 2^i$ be the binary expansion of j, and let $A_j = \{2^j\colon\ a_j = 1\}$. For each $A \subseteq [n]$ with Nim-sum 0, let $f(A) = A \Delta A_j$. Note that $f(A)$ has Nim-sum j. Since $(A\Delta A_j)\Delta A_j = A$, this map is a bijection into the set of subsets of $[n]$ with Nim-sum j.

Solved also by D. Beckwith, M. Benedicty, D. Berstein, J. C. Binz (Switzerland), M. Bowron, D. Callan, R. J. Chapman (U. K.), D. Donini (Italy), G. Gordon, R. Holzsager, K. S. Kedlaya, N. Komada, J. H. Lindsey II, J. Lorch, O. P. Lossers (The Netherlands), D. K. Nester, A. Nijenhuis, K. O'Bryant, M.-K. Siu (Hong Kong), J. H. Steelman, W. Stromquist, I. Vardi (Canada), H.-T. Wee (Singapore), M. Wolterman, Anchorage Math Solutions Group, GCHQ Problems Group (U. K.), NCCU Problems Group, NSA Problems Group, and the proposer.

Generalized Line Bingo

10565 [1997, 68]. *Proposed by D. M. Bloom, Brooklyn College, Brooklyn, NY, and Kenneth Suman, Winona State University, Winona, MN.* A rectangle is composed of mn squares arranged in m rows and n columns. In a certain game, the squares are selected one by one at random (without replacement). What is the expected number of selections until j columns of the rectangle are composed entirely of selected squares? (When $j = 1$, $m = 5$, and $n = 15$, this is the expected length of a type of bingo game known as a line game.)

Composite solution by the GCHQ Problems Group, Cheltenham, U. K. and the editors. For fixed m and n, the required expectation E_j equals $mn\prod_{i=j}^{n-1} mi/(mi+1)$.

For each instance of the game, we can continue selecting squares at random until all squares are selected. Thus it suffices to compute, over all permutations of the mn squares, the expected length of the initial segment that completes j columns. We compute for each square x the probability that it belongs to that initial segment. This is independent of x, so the expectation is mn times this probability.

Let A_i be the event that x belongs to the initial segment in which i columns are completed; note that $Pr(A_n) = 1$. The probability $Pr(A_j)$ is the product over $i \geq j$ of $Pr(A_i|A_{i+1})$.

We partition A_{i+1} into subevents that fix the trailing segment after the position where the $(i+1)$st column is completed. In such a subevent S, the identities of the first $i+1$ finished columns are fixed, but not which of these is last.

For permutations in S, let B be the set of squares consisting of the first i finished columns and the last square that completes the $(i+1)$st completed column. When $x \in B$, it is equally likely to occupy any of the $mi+1$ positions occupied by B, so the fraction of such permutations that belong to A_i is $mi/(mi+1)$.

When $x \notin B$, we can group the permutations by each fixed permutation of B. Now x is equally likely to fall into each of the $mi+1$ segments between members of B (or before the first). Again the fraction of these permutations that belong to A_i is $mi/(mi+1)$.

Editorial comment. The rows are unimportant. Víctor Hernández used linearity of expectation and the inclusion-exclusion principle to obtain a formula in the more general situation where the columns are sets of arbitrary size.

Solved also by R. J. Chapman (U. K.), D. A. Darling, V. Hernández (Spain), R. Holzsager, J. H. Lindsey II, P. W. Lindstrom, N. C. Singer, J. C. Smith, J. H. Steelman, Anchorage Math Solutions Group, and the proposer.

Ordered Trees and Stirling Numbers

10570 [1997, 69]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* An ordered tree is a rooted tree in which the children of each node form a sequence rather than a set. The height of an ordered tree is the number of edges on a path of maximum length starting at the root. Let $a(n,k)$ denote the number of ordered trees with n edges and height k, and let $S(n,k)$ be the Stirling number of the second kind (the number of partitions of $\{1,2,\ldots,n\}$ into k nonempty parts). Note that $a(n,1) = S(n,1)$, since both numbers are 1. Show that **(a)** $a(n,2) = S(n,2)$, **(b)** $a(n,3) + a(n,4) = S(n,3)$, and **(c)*** generalize these observations.

Solution I by Robin J. Chapman, University of Exeter, Exeter, UK. Let $b(n, k)$ be the number of ordered trees with n edges and height at most k. We include the tree with a root and no edges, so $b(0, k) = 1$ for $k \geq 0$. It suffices to show that $b(n, 2) = 1 + S(n, 2)$ and $b(n, 4) = 1 + S(n, 2) + S(n, 3)$ for $n > 0$. To achieve this, we compute the generating function $g_k(x) = \sum_{n\geq 0} b(n, k)x^n$ for $0 \leq k \leq 4$.

We have $g_0(x) = 1$. For $k > 0$, an ordered tree of height at most k consists of the root v, r edges incident to it, and a sequence of r ordered trees of height at most $k - 1$ rooted at the children of v. The generating function for ordered trees of height at most k in which the root has degree r is thus $x^r(g_{k-1}(x))^r$. Summing over r, we obtain $g_k(x) = 1/(1 - xg_{k-1}(x))$. Explicitly, this yields

$$g_1(x) = \frac{1}{1-x}, \qquad g_3(x) = \frac{1-2x}{1-3x+x^2},$$

$$g_2(x) = \frac{1-x}{1-2x}, \quad \text{and} \quad g_4(x) = \frac{1-3x+x^2}{1-4x+3x^2}.$$

Expanding by partial fractions yields

$$g_2(x) = 1 + \frac{x}{1-2x} \quad \text{and} \quad g_4(x) = 1 + \frac{x}{2(1-x)} + \frac{x}{2(1-3x)}.$$

Thus $b(n, 2) = 2^{n-1}$ and $b(n, 4) = (3^{n-1} + 1)/2$ for $n \geq 1$.

The number of partitions of $[n]$ into at most two parts is half the number of subsets of $[n]$, so $1 + S(n, 2) = 2^{n-1}$, as desired. Now consider partitions of $[n]$ into at most three parts. Each element other than n enters the part with n or one of the other two. Thus 3^{n-1} counts each partition with at least two parts twice, as we can interchange the second and third part without changing the partition. The partition with one part appears only once, so the total number of classes is $(3^{n-1} + 1)/2$, as desired.

Finding further relations among these numbers seem unlikely. If $f_k(x) = f_{k-1}(x) - xf_{k-2}(x)$ for $k \geq 2$, with $f_0(x) = f_1(x) = 1$, then $g_k(x) = f_k(x)/f_{k+1}(x)$ for all k. One can show that

$$f_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} x^j = \prod_{j=1}^{\lfloor k/2 \rfloor} \left(1 - 4x\cos^2\frac{j\pi}{k+1}\right).$$

It follows that

$$b(n, k) = \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} r_{k,j} \left(4\cos^2\frac{j\pi}{k+2}\right)^n$$

for constants $r_{k,j}$. On the other hand, it is well known that $S(n, k) = \sum_{j=1}^{k} t_{k,j} j^n$ for constants $t_{k,j}$. When $k \notin \{0, 1, 2, 4\}$, the value $4\cos^2 j\pi/(k + 2)$ is irrational for some j, so there is little hope of establishing a simple relationship betweeen $b(n, k)$ and the Stirling numbers in these cases.

Solution II to parts **(a)** *and* **(b)** *by Daniele Donini, Bertinoro, Italy.* We define a bijection from the set of ordered trees with n edges and height at most 4 to the set of partitions of $[n]$ with at most 3 blocks, in which for $k \leq 3$ the trees with height k become partitions with k blocks, and the trees with height 4 become partitions with 3 blocks.

Given a tree T, label the non-root vertices with the integers 1 through n in order via a depth-first left-first search. To form the corresponding partition, let the ith block consists of the label on vertices at distance i for the root, except as follows. In each subtree rooted at a vertex at distance 3 from the root, put the largest value in block 3 and put the other values in block 1. This reduces to partitioning by levels for trees with height 3.

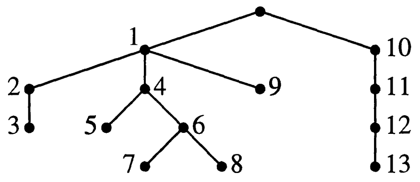
For the inverse map, index the blocks in partition π in increasing order of their least elements: A_1, possibly A_2, possibly A_3. Build the corresponding tree traversal (starting with 1 as the leftmost vertex at level 1) as follows. Suppose that the label just processed was label k, belonging to A_i. Let A_j be the block containing label $k+1$.

Case 1: $j \le i$. Make $k+1$ a new rightmost leaf at level j.

Case 2: $j = i+1$. Make $k+1$ the first child of the vertex with label k.

Case 3: $j = 3$ and $i = 1$. Because $\min A_2 < \min A_3$, there exists a label less than k in A_2. Let l be the largest such label less than k. Let m be the least label such that all labels from m to k lie in A_1; note that $m > l$. Remove $m, \dots, k$ from level 1. Make m the rightmost child of l (at level 3). Make $m+1, \dots, k+1$ children of m (at level 4).

Applying the original map to the resulting tree puts each label back into its block in π. As an example, the tree corresponding to $A_1 = \{1, 6, 7, 10, 12\}$, $A_2 = \{2, 4, 9, 11\}$, and $A_3 = \{3, 5, 8, 13\}$ is shown.



Solved also by D. Callan, R. Holzsager, the Anchorage Math Solutions Group, and the proposer.

Lattice Points Inside a Triangle

10600 [1997, 566]. *Proposed by Franz Rothe, University of North Carolina, Charlotte, NC.*
(a) Suppose a triangle has its vertices at integer lattice points in the plane and contains exactly 3 integer lattice points in its interior. Show that the center of mass of the triangle is not an integer lattice point.
(b)* Find all values i such that, if a triangle has its vertices at integer lattice points in the plane and contains exactly i integer lattice points in its interior, then the center of mass of the triangle cannot be an integer lattice point.

Solution of part **(a)** *by Robin J. Chapman, University of Exeter, Exeter, U. K.* Let the vertices of the triangle be A, B, and C, with position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, respectively. Suppose that the centroid of the triangle has integer coordinates. This centroid is $(1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Let D denote twice the area of the triangle. Then

$$D = |(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})| = |(\mathbf{c}-\mathbf{b}) \times (\mathbf{a}-\mathbf{b})| = |(\mathbf{a}-\mathbf{c}) \times (\mathbf{b}-\mathbf{c})| = |\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$$

where $\times$ denotes the vector product. By assumption $(1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c})$ has integer coordinates; therefore, so has $(1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c}) - \mathbf{a} = (1/3)\big((\mathbf{b}-\mathbf{a}) + (\mathbf{c}-\mathbf{a})\big)$. Hence $(\mathbf{c}-\mathbf{a}) = -(\mathbf{b}-\mathbf{a}) + 3\mathbf{d}$, where $\mathbf{d}$ has integer coordinates, and so $D = |(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})| = 3|(\mathbf{b}-\mathbf{a}) \times \mathbf{d}|$ is a multiple of 3. Let r, s, and t be the largest integers such that $(1/r)(\mathbf{b}-\mathbf{c})$, $(1/s)(\mathbf{c}-\mathbf{a})$, and $(1/t)(\mathbf{a}-\mathbf{b})$ have integer coordinates. Then the interiors of the sides BC, CA, and AB contain respectively $r-1$, $s-1$, and $t-1$ lattice points. By Pick's Theorem $D = 2N_1 + N_2 - 2$, where N_1 is the number of lattice points in the interior of the triangle and N_2 is the number of lattice points on its boundary (including the vertices). Consequently $D = 4 + r + s + t$. Also

$$D = st\left|\frac{\mathbf{b}-\mathbf{a}}{t} \times \frac{\mathbf{c}-\mathbf{a}}{s}\right|,$$

and so D is divisible by st. Similarly, D is divisible by rs and rt.

Since D is divisible by 3, we have $r+s+t \equiv 2 \pmod 3$. We now show that none of r, s, t is divisible by 3. Suppose that r is divisible by 3. Then $\mathbf{b}-\mathbf{c} = (\mathbf{b}-\mathbf{a}) + (\mathbf{a}-\mathbf{c}) = 3\mathbf{e}$