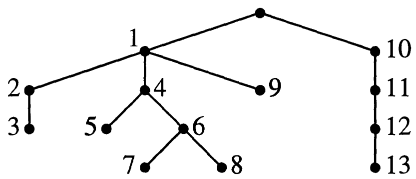
For the inverse map, index the blocks in partition π in increasing order of their least elements: A_1, possibly A_2, possibly A_3. Build the corresponding tree traversal (starting with 1 as the leftmost vertex at level 1) as follows. Suppose that the label just processed was label k, belonging to A_i. Let A_j be the block containing label $k+1$.
Case 1: $j \leq i$. Make $k+1$ a new rightmost leaf at level j.
Case 2: $j = i + 1$. Make $k+1$ the first child of the vertex with label k.
Case 3: $j = 3$ and $i = 1$. Because $\min A_2 < \min A_3$, there exists a label less than k in A_2. Let l be the largest such label less than k. Let m be the least label such that all labels from m to k lie in A_1; note that $m > l$. Remove $m, \dots, k$ from level 1. Make m the rightmost child of l (at level 3). Make $m+1, \dots, k+1$ children of m (at level 4).

Applying the original map to the resulting tree puts each label back into its block in π. As an example, the tree corresponding to $A_1 = \{1, 6, 7, 10, 12\}$, $A_2 = \{2, 4, 9, 11\}$, and $A_3 = \{3, 5, 8, 13\}$ is shown.



Solved also by D. Callan, R. Holzsager, the Anchorage Math Solutions Group, and the proposer.

Lattice Points Inside a Triangle

10600 [1997, 566]. *Proposed by Franz Rothe, University of North Carolina, Charlotte, NC.*
(a) Suppose a triangle has its vertices at integer lattice points in the plane and contains exactly 3 integer lattice points in its interior. Show that the center of mass of the triangle is not an integer lattice point.
(b)* Find all values i such that, if a triangle has its vertices at integer lattice points in the plane and contains exactly i integer lattice points in its interior, then the center of mass of the triangle cannot be an integer lattice point.

Solution of part **(a)** *by Robin J. Chapman, University of Exeter, Exeter, U. K.* Let the vertices of the triangle be A, B, and C, with position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, respectively. Suppose that the centroid of the triangle has integer coordinates. This centroid is $(1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Let D denote twice the area of the triangle. Then

$$D = |(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})| = |(\mathbf{c}-\mathbf{b}) \times (\mathbf{a}-\mathbf{b})| = |(\mathbf{a}-\mathbf{c}) \times (\mathbf{b}-\mathbf{c})| = |\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$$

where $\times$ denotes the vector product. By assumption $(1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c})$ has integer coordinates; therefore, so has $(1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c}) - \mathbf{a} = (1/3)\big((\mathbf{b}-\mathbf{a}) + (\mathbf{c}-\mathbf{a})\big)$. Hence $(\mathbf{c}-\mathbf{a}) = -(\mathbf{b}-\mathbf{a}) + 3\mathbf{d}$, where $\mathbf{d}$ has integer coordinates, and so $D = |(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})| = 3|(\mathbf{b}-\mathbf{a}) \times \mathbf{d}|$ is a multiple of 3. Let r, s, and t be the largest integers such that $(1/r)(\mathbf{b}-\mathbf{c})$, $(1/s)(\mathbf{c}-\mathbf{a})$, and $(1/t)(\mathbf{a}-\mathbf{b})$ have integer coordinates. Then the interiors of the sides BC, CA, and AB contain respectively $r-1$, $s-1$, and $t-1$ lattice points. By Pick's Theorem $D = 2N_1 + N_2 - 2$, where N_1 is the number of lattice points in the interior of the triangle and N_2 is the number of lattice points on its boundary (including the vertices). Consequently $D = 4 + r + s + t$. Also

$$D = st \left| \frac{\mathbf{b}-\mathbf{a}}{t} \times \frac{\mathbf{c}-\mathbf{a}}{s} \right|,$$

and so D is divisible by st. Similarly, D is divisible by rs and rt.

Since D is divisible by 3, we have $r + s + t \equiv 2 \pmod 3$. We now show that none of r, s, t is divisible by 3. Suppose that r is divisible by 3. Then $\mathbf{b}-\mathbf{c} = (\mathbf{b}-\mathbf{a}) + (\mathbf{a}-\mathbf{c}) = 3\mathbf{e}$

for some vector $\mathbf{e}$ with integer coordinates. But we already know that $(\mathbf{b}-\mathbf{a})-(\mathbf{a}-\mathbf{c}) = 3\mathbf{d}$. Hence both $\mathbf{b}-\mathbf{a}$ and $\mathbf{a}-\mathbf{c}$ are integer vectors multiplied by 3. Thus s and t are divisible by 3, contradicting $r+s+t \equiv 2 \pmod 3$. It follows that one of r, s, or t is congruent to 1 modulo 3, and the others are congruent to 2. Also D must be divisible by $3rs$, $3st$, and $3rt$.

We may assume that $r \geq s \geq t$. Now $3rs \leq D = 4+r+s+t$, and so

$$(3t-1)^2 \leq (3r-1)(3s-1) = 9rs - 3r - 3s + 1 \leq 13 + 3t.$$

This inequality is false for $t \geq 2$, so $t = 1$. Therefore $r \equiv s \equiv 2 \pmod 3$ and also $(3r-1)(3s-1) \leq 16$. Together, these imply that $r = s = 2$, and so $D = 9$. Now D is not divisible by $3rs$, so we have a contradiction.

Editorial comment. The proposer discovered the following result: Let a triangle have vertices at integer lattice points $(0,0)$, (b_1, b_2), and (c_2, c_2). Let $\alpha = \gcd(b_1 - c_2, b_2 - c_2)$, $\beta = \gcd(b_2, b_2)$, and $\gamma = \gcd(c_1, c_2)$. The center of mass is a lattice point if and only if either **(i)** 3 is a divisor of all three numbers α, β, and γ; or **(ii)** 3 is a divisor of none of the three numbers α, β, and γ, but 3 is a divisor of the double area $D = b_1c_2 - b_2c_1$.

Only partial solutions were received for part **(b)**. Searches by John H. Lindsey II and by the GCHQ Problems Group found the values $i < 1000$ satisfying the condition. The two lists are the same, except that 906 appears in one list and not the other. The remaining values found were: 3, 6, 15, 18, 30, 36, 48, 51, 63, 78, 90, 108, 120, 138, 150, 156, 168, 210, 228, 270, 300, 303, 336, 360, 378, 408, 426, 438, 480, 510, 528, 531, 630, 660, 723, 738, 750, 780, 888, 930, 990, 996.

Part **(a)** also solved by J. H. Lindsey II, GCHQ Problems Group (U. K.), and the proposer.

A Surrounded Set

10608 [1997, 664]. *Proposed by Victor Zalgaller, Steklov Mathematical Institute, St. Petersberg, Russia.* Let S be a compact convex set in the plane. If l is any line of support for S, let $f(l)$ be the length of the shortest curve that begins and ends on l and that together with l surrounds S. Prove that if $f(l)$ is independent of l, then S is a circle.

Solution by John Arkinstall, Monash University, Australia. Let l' be the support line parallel to l on the opposite side of S. Since $f(l) + f(l')$ is independent of l but also equals the perimeter of S plus twice the width of S perpendicular to l, S is a set of constant width w. A line l'' parallel to l and l' whose intersection with S is of maximum length is called a *diameter* of S in the direction of l. Because S has constant width w, we may conclude that the length of such a diameter is w, that it joins two points where support lines perpendicular to l touch S, that these two support lines, together with l and l', form a square of side w, and that each support line touches S in a unique point. A theorem of Khassa (Relation between maximal chords and symmetry for convex sets, *J. London Math. Soc.* **15** (1977) 541–546) states that a convex curve of constant width in which the diameter in each direction is midway between the two support lines in that direction must be a circle. Thus it suffices to prove this "midway" property.

When S is a set in the plane of constant width w, $2A(l) - wP(l)$ is independent of l, where $A(l)$ is the area of the portion of S between the opposite support line l' and the diameter l'', and $P(l)$ is the length of that part of the perimeter of S on the same side of l'' as l' (L. Beretta & A. Maxia, "Insiemi convessi e orbiformi," *Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat.* (5) **1** (1940) 1–64). Let $r(l)$ denote the distance from l to the diameter in the direction of l. Since $r(l)$ is the length of the supporting line segment from l to the boundary of S on the shortest curve from l surrounding S, we have $f(l) = 2r(l) + P(l)$. Since this is independent of l by hypothesis, so is $A(l) + wr(l)$. This is the area of the convex hull of S and the two supporting line segments from l on the shortest curve from l. The complement R of this convex hull in the supporting square to S with side length w and edge along l therefore also has area independent of the direction of l.