Now consider how the area of R changes when the direction of l is changed by a small angle ϕ. If the point of support on l' divides its side of the supporting square in the ratio $u : w - u$, then the area of R changes by four small approximately triangular regions: It decreases by $(1/2)r(l')\phi + o(\phi)$, increases by $(1/2)u\phi + o(\phi)$, decreases by $(1/2)(w - u)\phi + o(\phi)$, and increases by $(1/2)r(l')\phi + o(\phi)$. Thus, the area of R changes by the sum $(1/2)(2u - w)\phi + o(\phi)$. Since this is 0, we have $2u - w = 0$, and thus the support point on l' is midway between the two support lines perpendicular to l'.

Solved also by S. S. Kim (Korea), J. G. Merickel, GCHQ Problems Group (U. K.), and the proposer.

Tight Bounds for the Normal Distribution

10611 [1997, 665]. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{1 + \sqrt{1 - e^{-ax^2}}}{2} < \Phi(x) < \frac{1 + \sqrt{1 - e^{-bx^2}}}{2}$$

hold for all $x > 0$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. We show that $a = 1/2$ and $b = 2/\pi$ are the best possible constants for which the stated inequalities hold. Since $\int_{-\infty}^{0} e^{-y^2/2}\, dy = \int_{0}^{\infty} e^{-y^2/2}\, dy = \sqrt{\pi/2}$, the stated inequalities are equivalent to

$$\frac{\sqrt{1 - e^{-ax^2}}}{2} < f(x) < \frac{\sqrt{1 - e^{-bx^2}}}{2}, \tag{1}$$

where $f(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-y^2/2}\, dy$. If the second inequality of (1) holds for all $x > 0$, then

$$0 < \frac{\sqrt{1 - e^{-bx^2}}}{2} - f(x) = \left(\frac{\sqrt{b}}{2} - \frac{1}{\sqrt{2\pi}}\right) x + O(x^3)$$

as $x \to 0$, which implies $b \geq 2/\pi$. Similarly, if the first inequality of (1) holds for all $x > 0$, then

$$0 < f(x) - \frac{\sqrt{1 - e^{-ax^2}}}{2} = \frac{1}{4} e^{-ax^2} + O\left(e^{-2ax^2}\right) - \frac{e^{-x^2/2}}{\sqrt{2\pi}x} + O\left(\frac{e^{-x^2/2}}{x^2}\right)$$

as $x \to \infty$. Dividing each side by $e^{-x^2/2}$ yields $a \leq 1/2$.

To show that inequalities (1) hold for all $x > 0$ when $a = 1/2$ and $b = 2/\pi$, we write

$$(f(x))^2 = \frac{1}{2\pi} \int_0^x \int_0^x e^{-(y^2+z^2)/2}\, dy dz.$$

Let $D = [0, x]^2$, $D_1 = \{ (y, z) : 0 \leq y,\ 0 \leq z,\ y^2 + z^2 \leq x^2 \}$, and $D_2 = \{ (y, z) : 0 \leq y,\ 0 \leq z,\ y^2 + z^2 \leq (4/\pi)x^2 \}$. We have the inequalities

$$\frac{1}{2\pi} \iint_{D_1} e^{-(y^2+z^2)/2}\, dydz < \frac{1}{2\pi} \iint_{D} e^{-(y^2+z^2)/2}\, dydz < \frac{1}{2\pi} \iint_{D_2} e^{-(y^2+z^2)/2}\, dydz, \tag{2}$$

the first because $D_1 \subset D$, and the second because D and D_2 have the same area and $e^{-(y^2+z^2)/2} \leq e^{-(2/\pi)x^2}$ for $(y, z) \in D - D_2$ while $e^{-(y^2+z^2)/2} \geq e^{-(2/\pi)x^2}$ for $(y, z) \in D_2 - D$. Evaluating the outer integrals in (2) in polar coordinates, we obtain

$$\frac{1 - e^{-x^2/2}}{4} < f(x)^2 < \frac{1 - e^{-2x^2/\pi}}{4},$$

which is equivalent to (1).

Solved also by P. Alsholm (Denmark), J. Anglesio (France), P. Bracken (Canada), B. Burdick, G. G. Chappell, P. Devaraj (India), G. Keselman, K.-W. Lau (Hong Kong), J. H. Lindsey II, A. Stadler, GCHQ Problems Group (U. K.), NCCU Problems Group, NSA Problems Group, WMC Problems Group, and the proposer.