



Tight Bounds for the Normal Distribution: 10611

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The American Mathematical Monthly, Vol. 106, No. 1. (Jan., 1999), p. 76.

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Now consider how the area of R changes when the direction of l is changed by a small angle ϕ . If the point of support on l' divides its side of the supporting square in the ratio $u : w - u$, then the area of R changes by four small approximately triangular regions: It decreases by $(1/2)r(l')\phi + o(\phi)$, increases by $(1/2)u\phi + o(\phi)$, decreases by $(1/2)(w - u)\phi + o(\phi)$, and increases by $(1/2)r(l')\phi + o(\phi)$. Thus, the area of R changes by the sum $(1/2)(2u - w)\phi + o(\phi)$. Since this is 0, we have $2u - w = 0$, and thus the support point on l' is midway between the two support lines perpendicular to l' .

Solved also by S. S. Kim (Korea), J. G. Merickel, GCHQ Problems Group (U. K.), and the proposer.

Tight Bounds for the Normal Distribution

10611 [1997, 665]. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{1 + \sqrt{1 - e^{-ax^2}}}{2} < \Phi(x) < \frac{1 + \sqrt{1 - e^{-bx^2}}}{2}$$

hold for all $x > 0$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. We show that $a = 1/2$ and $b = 2/\pi$ are the best possible constants for which the stated inequalities hold. Since $\int_{-\infty}^0 e^{-y^2/2} dy = \int_0^{\infty} e^{-y^2/2} dy = \sqrt{\pi/2}$, the stated inequalities are equivalent to

$$\frac{\sqrt{1 - e^{-ax^2}}}{2} < f(x) < \frac{\sqrt{1 - e^{-bx^2}}}{2}, \quad (1)$$

where $f(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2} dy$. If the second inequality of (1) holds for all $x > 0$, then

$$0 < \frac{\sqrt{1 - e^{-bx^2}}}{2} - f(x) = \left(\frac{\sqrt{b}}{2} - \frac{1}{\sqrt{2\pi}} \right) x + O(x^3)$$

as $x \rightarrow 0$, which implies $b \geq 2/\pi$. Similarly, if the first inequality of (1) holds for all $x > 0$, then

$$0 < f(x) - \frac{\sqrt{1 - e^{-ax^2}}}{2} = \frac{1}{4} e^{-ax^2} + O(e^{-2ax^2}) - \frac{e^{-x^2/2}}{\sqrt{2\pi x}} + O\left(\frac{e^{-x^2/2}}{x^2}\right)$$

as $x \rightarrow \infty$. Dividing each side by $e^{-x^2/2}$ yields $a \leq 1/2$.

To show that inequalities (1) hold for all $x > 0$ when $a = 1/2$ and $b = 2/\pi$, we write

$$(f(x))^2 = \frac{1}{2\pi} \int_0^x \int_0^x e^{-(y^2+z^2)/2} dy dz.$$

Let $D = [0, x]^2$, $D_1 = \{(y, z) : 0 \leq y, 0 \leq z, y^2 + z^2 \leq x^2\}$, and $D_2 = \{(y, z) : 0 \leq y, 0 \leq z, y^2 + z^2 \leq (4/\pi)x^2\}$. We have the inequalities

$$\frac{1}{2\pi} \iint_{D_1} e^{-(y^2+z^2)/2} dy dz < \frac{1}{2\pi} \iint_D e^{-(y^2+z^2)/2} dy dz < \frac{1}{2\pi} \iint_{D_2} e^{-(y^2+z^2)/2} dy dz, \quad (2)$$

the first because $D_1 \subset D$, and the second because D and D_2 have the same area and $e^{-(y^2+z^2)/2} \leq e^{-(2/\pi)x^2}$ for $(y, z) \in D - D_2$ while $e^{-(y^2+z^2)/2} \geq e^{-(2/\pi)x^2}$ for $(y, z) \in D_2 - D$. Evaluating the outer integrals in (2) in polar coordinates, we obtain

$$\frac{1 - e^{-x^2/2}}{4} < f(x)^2 < \frac{1 - e^{-2x^2/\pi}}{4},$$

which is equivalent to (1).

Solved also by P. Alsholm (Denmark), J. Anglesio (France), P. Bracken (Canada), B. Burdick, G. G. Chappell, P. Devaraj (India), G. Keselman, K.-W. Lau (Hong Kong), J. H. Lindsey II, A. Stadler, GCHQ Problems Group (U. K.), NCCU Problems Group, NSA Problems Group, WMC Problems Group, and the proposer.