Trigonometric Integrals and Hadamard Products

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1. INTRODUCTION. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic functions in respective disks D_1 and D_2 centered at the origin. Let $h(\xi) = \sum_{n=0}^{\infty} a_n b_n \xi^n$ be analytic in some disk centered at $\xi = 0$. In a famous 1899 paper [**5**], J. Hadamard used the integral convolution

$$h(\xi) = (2\pi)^{-1} \int_0^{2\pi} f(ze^{i\theta}) g(\zeta e^{-i\theta})\, d\theta \tag{1.1}$$

($\xi = z\zeta$) to characterize the singularities of the function $h(\xi)$ in terms of those of $f(z)$ and $g(z)$ (see also [**13**, p. 157]). The coefficient-wise product of power series denoted by $h(\xi) = f(z) \circ g(\zeta)$ is often called the *Hadamard product* of $f(z)$ and $g(\zeta)$ (alternative appellations include *Schur product* and *quasi inner product*).

The Hadamard product appears naturally in a variety of theoretical and applied mathematical questions. While studying the Bieberbach conjecture long before L. deBrange's proof, C. Loewner and E. Netanyahu [7] showed that a Hadamard-type product of two normalized ($f(0) = 0$, $f'(0) = 1$) univalent (one-to-one) analytic functions in the unit disk need not be univalent and can even have a zero derivative. St. Ruschewegh and T. Sheil-Small [**12**] showed that the Hadamard product of two normalized analytic convex mappings of the unit disk is a convex mapping. In [**3**], the Hadamard product appears in a variety of multiplier problems. In an 1894 paper [**9**], Th. Moutard used (essentially) the fact that an entry-wise product of positive definite matrices is positive definite to establish uniqueness theorems for solutions of a class of elliptic partial differential equations. Many properties and applications of the Hadamard product are surveyed in [**6**]. The author has used this product and a generalization to derive known and new formulas for special functions [**1**]. Finally, the Hadamard product, together with a factor switching property, was used in [**2**] to construct solution formulas for a variety of Cauchy-type problems.

Because of its connection with the integral convolution (1.1), the Hadamard product can lead to elegant evaluations of complicated trigonometric integrals and provide analytic derivations of combinatorial identities. The student with a solid background in the calculus and basic differential equations can use it to carry out beginning studies on special functions and their representations while being introduced to notions of complex variables. We focus on these areas while emphasizing how to choose the functions in the associated Hadamard products.

For example, consider the series definition for the Bessel function

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} / \{2^{2n} n! \cdot n!\} = \sum_{n=0}^{\infty} (1/(2^n n!))((-1)^n/(2^n n!)) z^{2n}$$

and observe that the terms $1/(2^n n!)$ are the coefficients in the series for $e^{z/2}$ while the terms $(-1)^n/(2^n n!)$ are the coefficients in the series for $e^{-z/2}$. It follows that

$$J_0(z) = e^{z/2} \circ e^{-z/2} = (2\pi)^{-1} \int_0^{2\pi} e^{(ze^{i\theta})/2} e^{-(ze^{i\theta})/2}\, d\theta.$$

Applying the Euler formulas to the integrand yields the standard integral formula

$$J_0(z) = (2\pi)^{-1} \int_0^{2\pi} \cos(z \sin\theta)\, d\theta.$$

Before outlining the topics to be considered in the following sections, we introduce a Hadamard-type product depending on two integer parameters.

Definition 1. *For p and q relatively prime, and for $z \in D_1$, $\zeta \in D_2$, define*

$$f(z)_p \circ_q g(\zeta) = (2\pi)^{-1} \int_0^{2\pi} f(ze^{pi\theta}) g(\zeta e^{-qi\theta})\, d\theta = \sum_{n=0}^{\infty} a_{qn} b_{pn} z^{qn} \zeta^{pn}. \quad (1.2)$$

If either $f(z)$ or $g(\zeta)$ is a polynomial, then the series in (1.1) *is a polynomial in $z\zeta$.*

In Section 2, *we apply* (1.1) *in various ways to polynomials of the form* $(1 \pm z)^n$ *to obtain evaluations of numerous trigonometric integrals and prove binomial identities. Repeated Hadamard products are employed in Section* 3 *to evaluate multiple trigonometric integrals. Section* 4 *treats integral representations of some special functions in which at least one of the functions entering the associated Hadamard product is not a polynomial. Finally, in Section* 5, *we introduce the selector function, which sums the coefficients of a subseries of a given series, and show how it can be used to replace certain sums by integrals.*

2. SPECIAL SUMS AND SINGLE INTEGRALS. *Let us now use* (1.1) *to evaluate some special sums and trigonometric integrals.*

Example 2.1. *We wish to establish the familiar binomial identity*

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j}^2 = (-1)^n \binom{2n}{n} \quad (2.1)$$

(see **[11]** *for this and related identities). On the one hand, through integration by parts, one obtains a reduction relation which permits showing*

$$\int_0^{2\pi} \sin^{2n}\theta\, d\theta = \frac{2\pi}{2^{2n}} \binom{2n}{n}. \quad (2.2)$$

Alternatively, we note that $2i\sin\theta = e^{i\theta} - e^{-i\theta} = (1 + e^{i\theta})(1 - e^{-i\theta})$, *which suggests considering the Hadamard product* $(1+z)^{2n} \circ (1-z)^{2n}$. *With* $a_j = \binom{2n}{j}$, $b_j = (-1)^j a_j$, *and* $p = q = 1$, (1.1) *gives*

$$\begin{aligned}(1+z)^{2n} \circ (1-z)^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} (1 + ze^{i\theta})^{2n} (1 - ze^{-i\theta})^{2n}\, d\theta \\ &= \sum_{j=0}^{2n} (-1)^j \binom{2n}{j}^2 z^{2j}. \quad (2.3)\end{aligned}$$

Setting $z = 1$, *we have*

$$\int_0^{2\pi} \sin^{2n}\theta \, d\theta = (-1)^n \frac{2\pi}{2^{2n}} \sum_{j=0}^{2n} (-1)^j \binom{2n}{j}^2, \tag{2.4}$$

which, together with (2.2) *gives* (2.1).

Example 2.2. *Suppose that p and q are relatively prime and let r be a positive integer. We prove*

$$\int_0^{\pi} \cos^{pr}(q\theta) \cdot \cos^{qr}(p\theta) \, d\theta = \frac{\pi}{2^{r(p+q)}} \sum_{l=0}^{r} \binom{qr}{ql} \binom{pr}{pl}. \tag{2.5}$$

Replace $\cos\theta$ *by* $(e^{i\theta} + e^{-i\theta})/2$ *and take* $f(z) = (1+z)^{qr}$ *and* $g(z) = (1+z)^{pr}$ *in* (1.2), *which becomes*

$$\begin{aligned} f(z)_p \circ {}_q g(z) &= (2\pi)^{-1} \int_0^{2\pi} (1 + ze^{pi\theta})^{qr} (1 + ze^{-qi\theta})^{pr} \, d\theta \\ &= \sum_{l=0}^{r} \binom{qr}{ql} \binom{pr}{pl} z^{l(p+q)}. \end{aligned} \tag{2.6}$$

Compute

$$\begin{aligned} &(1 + e^{pi\theta})^{qr} (1 + e^{-qi\theta})^{pr} \\ &= \left[e^{pqri\theta/2} (e^{pi\theta/2} + e^{-pi\theta/2})^{qr} \right] \left[e^{-qpri\theta/2} (e^{qi\theta/2} + e^{-qi\theta/2})^{pr} \right] \\ &= \left(2\cos\frac{p\theta}{2} \right)^{qr} \left(2\cos\frac{q\theta}{2} \right)^{pr} = 2^{r(p+q)} \cos^{qr}\left(\frac{p\theta}{2} \right) \cos^{pr}\left(\frac{q\theta}{2} \right), \end{aligned}$$

insert this into the integral in (2.6) (*with* $z = 1$), *make the change of variables* $\theta = 2\phi$, *and* (2.5) *follows.*

A generalization of (2.5) *can be obtained by taking* $f(z) = (1+z)^m$ *and* $g(z) = (1+z)^n$. *By applying* (1.1), *taking the real part of the integral involved, and making a simple change of variables, one can establish*

$$\int_0^{\pi} (\cos p\phi)^m (\cos q\phi)^n \cos(mp - nq)\phi \, d\phi = \frac{\pi}{2^{m+n}} \sum_{l=0}^{H} \binom{m}{ql} \binom{n}{pl} \tag{2.7}$$

where $H = \min\left\{ \left[\frac{m}{q} \right], \left[\frac{n}{p} \right] \right\}$. *A variety of analogous integrals appear in* [**4**, *pp.* 372–374].

Example 2.3. *Let* $z_1 = z_2 = z$ *and take* $f(z) = g(z) = (a + bz + az^2)^n$. *The trinomial expansion gives* $f(z) = \sum_{k=0}^{2n} c_k z^k$, *in which*

$$c_k = \sum_{j_2 + 2j_3 = k} \binom{n}{j_1, j_2, j_3} a^{j_1 + j_3} b^{j_2}$$

where $j_1 + j_2 + j_3 = n$ *and* $j_i \geq 0$. *We show that*

$$\sum_{k=0}^{2n} c_k^2 = b^{2n} {}_2F_1(-n, -n + 1/2; 1; 4a^2/b^2). \tag{2.8}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ *denotes the usual Gauss hypergeometric function with parame-*

ters α, β, and γ [**8**, *pp.* 37–62]. *Compute*

$$\sum_{k=0}^{2n} c_k^2 = (f(z) \circ f(z))|_{z=1}$$

$$= (2\pi)^{-1} \int_0^{2\pi} (a + be^{i\theta} + ae^{2i\theta})^n (a + be^{-i\theta} + ae^{-2i\theta})^n \, d\theta$$

$$= (2\pi)^{-1} \int_0^{2\pi} (2a\cos\theta + b)^{2n} \, d\theta. \tag{2.9}$$

Expand the integrand in the latter integral, integrate term-by-term, and use the beta function and its reduction properties (*see* [**8**, *pp.* 7–8]) *to find that*

$$\sum_{k=0}^{2n} c_k^2 = \sum_{k=0}^{n} \frac{(2n)! \cdot a^{2j} b^{2n-2j}}{(2n-2k)!k!k!}. \tag{2.10}$$

Finally, replace $(2n)!/(2n-2k)!$ *in this by* $2^{2k}(-n)_k(-n+1/2)_k$ *to obtain* (2.8).

3. SOME MULTIPLE TRIGONOMETRIC INTEGRALS. *The Hadamard product of two polynomials is a new polynomial, which one can use to form a Hadamard product with another polynomial, etc. By repeating the Hadamard product operation and finally evaluating at* $z = 1$, *as in Section* 2, *one can evaluate various multiple trigonometric integrals.*

Example 3.1. *As in* (2.1.6), *we have*

$$(1+z)^n \circ (1+z)^n = (2\pi)^{-1} \int_0^{2\pi} (1 + ze^{i\theta})^n (1 + ze^{-i\theta})^n d\theta$$

$$= \sum_{j=0\}}^{n} \binom{n}{j}^2 z^{2j} = F(z). \tag{3.1}$$

An application of (1.1) *to* $F(z) \circ (1+z)^n$ *yields*

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \left[(1 + ze^{i(\phi+\theta)})(1 + ze^{i(\phi-\theta)})(1 + ze^{-i\phi})\right]^n d\phi \, d\theta$$

$$= \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{j}^2 \binom{n}{2j} z^{4j}. \tag{3.2}$$

Set $z = 1$ *and observe that the bracketed term in the integrand can be written as* $2^3 e^{i\phi/2} \cos((\phi+\theta)/2) \cdot \cos((\phi-\theta)/2)\cos(\phi/2) = 2^2 e^{i\phi/2}\cos(\phi/2)[\cos\phi + \cos\theta]$. *Insert this into* (3.2), *with* $z = 1$, *and take the real part of both sides to get*

$$\int_0^{2\pi} \int_0^{2\pi} \cos(n\phi/2)\cos^n(\phi/2)[\cos\phi + \cos\theta]^n d\phi \, d\theta = \frac{4\pi^2}{2^{2n}} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{j}^2 \binom{n}{2j}. \tag{3.3}$$

A similar treatment of $G(z) = F(z) \circ F(z)$ leads to the triple integral formula

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}[(\cos\theta_1 + \cos\theta_2)(\cos\theta_1 + \cos\theta_3)]^n d\theta_1\, d\theta_2\, d\theta_3 = \frac{(2\pi)^3}{2^{2n}}\sum_{k=0}^{n}\binom{n}{k}^4. \tag{3.4}$$

Example 3.2. *The repeated Hadamard product $[(1+z)^m \circ (1+z)^n] \circ (1+z^2)^p$ yields the double integral formula*

$$\int_0^{\pi}\int_0^{\pi}\cos[(m-n)\theta + (m+n-2p)\phi]\cos^m(\theta+\phi)\cos^n(\theta-\phi)\cos^p(2\phi)\,d\theta\,d\phi$$

$$= \frac{\pi^2}{2^{m+n+p}}\sum_{j=0}^{\min(m,n,p)}\binom{m}{j}\binom{n}{j}\binom{p}{j}. \tag{3.5}$$

Taking $m = n = p$, it is not difficult to establish

$$\int_0^{2\pi}\int_0^{2\pi}[\cos\theta(\cos\theta + \cos\phi)]^m d\theta\,d\phi = \frac{4\pi^2}{2^{2m}}\sum_{j=0}^{m}\binom{m}{j}^3.$$

4. SOME SPECIAL POLYNOMIALS. *Thus far, both Hadamard product factors $f(z)$ and $g(\zeta)$ have been polynomials. A convenient non-polynomial choice for $f(z)$ is the exponential function and we shall use it to obtain trigonometric integral representations for the classical Laguerre polynomials and for some special hypergeometric type polynomials.*

Example 4.1. *The Laguerre Polynomials. The Laguerre polynomials are defined by $L_n(z) = \sum_{j=0}^{n}\frac{(-1)^j}{j!}\cdot\binom{n}{j}z^j$ (see* [**8**, *p.* 239] *or* [**10**, *pp.* 200–217]). *We identify $\binom{n}{j}$ as the Maclaurin coefficients of $(1+\zeta)^n$ and $(-1)^j/j!$ as the Maclaurin coefficients of e^{-z}. Then using* (1.1), *we have*

$$L_n(z\zeta) = \sum_{j=0}^{n}\left\{\frac{(-1)^j z^j}{j!}\right\}\left\{\binom{n}{j}\zeta^j\right\} = e^{-z}\circ(1+\zeta)^n$$

$$= \frac{1}{2\pi}\int_0^{2\pi}e^{-ze^{i\theta}}\left(1+\zeta e^{-i\theta}\right)^n d\theta. \tag{4.1}$$

If we now select $z = x$ (real) and $\zeta = 1$, we obtain

$$L_n(x) = \operatorname{Re}\left\{(2\pi)^{-1}\int_0^{2\pi}e^{-x\cos\theta}\cdot e^{-ix\sin\theta - n\theta i/2}\left(e^{i\theta/2} + e^{-i\theta/2}\right)^n d\theta\right\}$$

$$= 2^n(2\pi)^{-1}\int_0^{2\pi}e^{-x\cos\theta}\cos(x\sin\theta + n\theta/2)\cos^n(\theta/2)\,d\theta$$

$$= 2^n\pi^{-1}\int_0^{\pi}e^{-x\cos 2\phi}\cos(x\sin 2\phi + n\phi)\cos^n\phi\,d\phi. \tag{4.2}$$

Since $L_n(0) = 1$, this yields $\int_0^{\pi}\cos(n\phi)\cos^n\phi\,d\phi = \pi/2^n$.

Example 4.2. *A Hypergeometric Polynomial. Select* $f(z) = e^{-z}$ *and* $g(\zeta) = (1-\zeta)^n$. *Then* (1.1) *gives*

$$e^{-z}{}_1 \circ {}_2(1-\zeta)^n = (2\pi)^{-1}\int_0^{2\pi} e^{-ze^{2i\theta}}\left(1-\zeta e^{-i\theta}\right)^n d\theta$$
$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!}\binom{n}{2k} z^k\zeta^k = p_n(z,\zeta). \qquad (4.3)$$

Choose $z = x$, x *real, and* $\zeta = 1$ *and obtain*

$$p_n(x,1) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!}\frac{n!}{(2k)!(n-2k)!}x^k.$$

Taking the real and imaginary parts of the integral in (4.3), *and making the change of variables* $\theta = 2\phi$ *gives*

$$p_n(x,1) = \begin{cases} \dfrac{(-1)^{n/2}2^n}{\pi}\displaystyle\int_0^{\pi} e^{-x\cos 4\phi}\cos(x\sin 4\phi + n\phi)\sin^n\phi\, d\phi & \text{for } n \text{ even,} \\ \dfrac{(-1)^{(n-1)/2}2^n}{\pi}\displaystyle\int_0^{\pi} e^{-x\cos 4\phi}\sin(x\sin 4\phi + n\phi)\sin^n\phi\, d\phi & \text{for } n \text{ odd.} \end{cases}$$
$$(4.4)$$

5. THE SELECTOR FUNCTION. *Let* $S_k(z) = \sum_{j=0}^{k} z^j = (1-z^{k+1})/(1-z)$ *and let* $P_n(z) = \sum_{j=0}^{n} a_j z^j$. *Form* $P_n(z)_1 \circ {}_q S_k(z)$ *and replace* z *by* 1 *to obtain*

$$(2\pi)^{-1}\int_0^{2\pi} P_n(e^{i\theta})\frac{\left(1-e^{-q(k+1)i\theta}\right)}{\left(1-e^{-qi\theta}\right)}d\theta =$$

$$(2\pi)^{-1}\int_0^{2\pi} P_n(e^{i\theta})e^{-qki\theta/2}\frac{\sin q(k+1)\theta/2}{\sin q\theta/2}d\theta = \sum_{j=0}^{\min\left\{\left[\frac{n}{q}\right],\left[\frac{k}{q}\right]\right\}} a_{qj}. \qquad (5.1)$$

The latter series is a sum of a subset of the coefficients of the polynomial $P_n(z)$ *in which the indices are multiples of* q. *This particular subset also depends upon the choice of* k. *It is clear that if* $q = 1$ *and* $k \geq n$, *then the series in* (5.1) *reduces to* $P(1)$. *Thus, with appropriate choices of* q *and* k, (5.1) *can be applied in a variety of ways to replace finite sums by trigonometric integrals. Because of its utility for singling out particular terms of a polynomial, we refer to* $S_k(z)$ *as a selector function.*

To write an integral formula for the binomial sum $\sum_{j=0}^{m}\binom{n}{j}$, take $P_n(z) = (1+z)^n$, $q = 1$, and $k = m$ in (5.1). Applying the Euler relations and taking $\theta = 2\phi$ leads to

$$\sum_{j=0}^{m}\binom{n}{j} = \frac{2^n}{\pi}\int_0^{\pi}\cos^n(\phi)\cos((n-m)\phi)\frac{\sin((m+1)\phi)}{\sin(\phi)}d\phi. \qquad (5.2)$$

While the denominator in the integrand vanishes at $\phi = 0$ and $\phi = \pi$, so also does $\sin((m+1)\phi)$. The singularities cancel and the integral (5.2) is well defined.

As a final example, take $P_n(z) = (1 + z)^{pq}$ and $k = p$ in (5.1) with $q \geq 1$. We leave it to the reader to conclude that

$$\int_0^{\pi} \cos^{pq}\phi \frac{\sin q(p+1)\phi}{\sin q\phi} \, d\phi = \frac{2\pi}{2^{pq}} \sum_{k=0}^{p} \binom{pq}{kq}. \tag{5.3}$$

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