NOTES

Edited by **Jimmie D. Lawson and William Adkins**

Fermat's Last Theorem for Gaussian Integer Exponents

John A. Zuehlke

In this note, we observe that Wiles' Theorem [**2**] on the impossibility of

$$x^n + y^n = z^n$$

for x, y, z positive rational numbers with integer exponents $n \neq \pm 1, \pm 2$ can be generalized to the case of Gaussian integer exponents $\nu = n + im$ without additional exceptions. The proof uses the Gelfond-Schneider Theorem [**1**], according to which α^{β} is transcendental for β algebraic but not rational and α algebraic $\neq 0, 1$.

The proof almost fits into the margin. In fact, from

$$x^{\nu} + y^{\nu} = z^{\nu}, \text{with } \nu = n + im, \quad m \neq 0$$

it follows by taking the complex modulus squared that

$$x^{2n} + 2x^n y^n \cos\theta + y^{2n} = z^{2n}, \text{with } \theta = m\log(x/y),$$

so $\cos\theta$ is rational. Since, for any real number θ whatsoever there is the identity

$$e^{2i\theta} - 2\cos\theta e^{i\theta} + 1 = 0,$$

it follows for the particular θ that

$$e^{i\theta} = (x/y)^{im}$$

is algebraic. Then the Gelfond-Schneider Theorem, with $\alpha = x/y$ and $\beta = im$ forces $x = y$. Therefore

$$(z/x)^{\nu} = 2,$$

forcing $z = x$ similarly, contradicting $y \neq 0$.

We remark that the generalization holds, with the same proof, for exponents $\nu = n + im$, with n an integer and m a real algebraic number.

REFERENCES

1. A. Baker, *Transcendental Number Theory*, Cambridge University Press (2nd edition), Cambridge 1979.
2. A. Wiles, Modular Elliptic Curves and Fermat's Last Theorem, *Annals of Math.* **141** (1995) 443–551.

Columbia University, New York, New York 10027
jaz@cpw.math.columbia.edu