

On Rational Function Approximations to Square Roots

M. J. Jamieson

The American Mathematical Monthly, Vol. 106, No. 1. (Jan., 1999), pp. 50-52.

Stable URL:

http://links.jstor.org/sici?sici=0002-9890%28199901%29106%3A1%3C50%3AORFATS%3E2.0.CO%3B2-W

The American Mathematical Monthly is currently published by Mathematical Association of America.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <u>http://www.jstor.org/journals/maa.html</u>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.

On Rational Function Approximations to Square Roots

M. J. Jamieson

Interest in methods for calculating square roots exists partly because the speed with which a computer can evaluate them contributes a measure of its overall performance [1]. Newton's well known method for improving estimates of a square root uses a simple rational function approximation and converges in second order; an iterative method converges in second order if it generates a sequence $\{s_n\}$ that tends to limit s and, in which the error $(s_{n+1} - s)$ tends to $K(s_n - s)^2$ for some K, independent of n, as n tends to infinity. This note presents a rational function approximation method with faster convergence. It uses a result of Frank [2] on periodic continued fractions; a formula given in the 15th century by Al-Kalsadi [3, p. 111] is also based on continued fractions and is a special case.

Frank studied properties of the convergents (or approximants) to the pure periodic continued fraction representing the quadratic surd

$$L + \sqrt{C} = \left[b_0, \overline{b_1, \dots, b_p}\right] = b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots,$$
(1)

where L and C are rational numbers with C positive, the b's are positive integers, p is the period and the overbar indicates the periodic part of the continued fraction. There are restrictions on the values of L and C in order that the representing continued fraction be pure periodic [4, p. 101], but they are satisfied if L is zero and C is an integer (when $b_p = 2b_0$, although this is not important here). A special case of a property given by Frank ([2, (2.2)] with s = 1, C = N, L = 0) is

$$x_{i+j} = (x_i x_j + N) / (x_i + x_j) \quad \text{for } i, j > 0,$$
(2)

where x_i is the (pi - 1)th convergent to the continued fraction. It can be shown by induction, from Pascal's triangle rule for binomial coefficients $\binom{i}{j}$, that the convergents satisfy

$$x_k = F_k(x_1) \quad \text{for } k > 0, \tag{3}$$

where, if k is odd (= 2m + 1),

$$F_{k}(x) = \sum_{i=0}^{i=m} \binom{k}{k-2i} x^{k-2i} N^{i} \div \sum_{i=0}^{i=m} \binom{k}{k-2i-1} x^{k-2i-1} N^{i}$$

and, if k is even (= 2m),

$$F_{k}(x) = \sum_{i=0}^{i=m} \binom{k}{k-2i} x^{k-2i} N^{i} \div \sum_{i=0}^{i=m-1} \binom{k}{k-2i-1} x^{k-2i-1} N^{i}.$$
 (4)

The function $F_k(x)$ has a fixed point at \sqrt{N} . Equation (3) gives x_k in terms of x_1 . By considering the period of the continued fraction to be the multiple $k^n p$ of p instead of p itself we obtain the same formula giving $x_{k^{n+1}}$ in terms of x_{k^n} . Thus function (4) generates a series of approximations to \sqrt{N} which form a subsequence of the convergents to the continued fraction if one starts with $x = x_1$. We can use function (4) in an iterative scheme for finding the square root of an integer. To find the square root of a rational number q/r, say, we calculate \sqrt{qr} and divide by r.

The theory of continued fractions guarantees convergence (unless k = 1) but we must know the value of the (p - 1)th convergent to start the iterative sequence. This is inconvenient. However, by the following theorem, convergence is also guaranteed if we start with an arbitrary positive value x.

Theorem. For x > 0, $y_n := F_{K^n}(x) \to \sqrt{N}$.

We generalize (2) and (3) and replace (2) by

$$F_{i+j}(x) = [F_i(x)F_j(x) + N] \div [F_i(x) + F_j(x)] \text{ for } i, j > 0.$$
(5)

From (3), (4), and (5) we find

$$F_{k}(x) - \sqrt{N} = (x - \sqrt{N}) \left[F_{k-1}(x) - \sqrt{N} \right] \div \left[x + F_{k-1}(x) \right] \text{ for } k > 1.$$
(6)

From the definition of $F_k(x)$, if x is positive so is $F_k(x)$ for any k. By induction from (3), (4), and (6) we find

$$F_k(x) \ge \sqrt{N} \quad \text{if} \quad x \ge \sqrt{N} \,,$$
 (7a)

$$F_k(x) < x$$
 if $x > \sqrt{N}$. (7b)

These inequalities imply that repeated application of the function $F_k(x)$ generates a strictly monotonic decreasing sequence whose greatest lower bound is \sqrt{N} if the starting value exceeds \sqrt{N} . Hence convergence is guaranteed for any starting value exceeding \sqrt{N} . An argument similar to that leading to inequality (7a) shows that

$$F_k(x) \ge \sqrt{N}$$
 if $x \le \sqrt{N}$ for k even, (8a)

$$F_k(x) \le \sqrt{N}$$
 if $x \le \sqrt{N}$ for k odd. (8b)

For even k and starting value smaller that \sqrt{N} the first generated value exceeds \sqrt{N} and convergence is guaranteed by the argument of the preceding paragraph. If k is odd it can be shown from (6) and (8) that the function $F_k(x)$ is strictly monotonic increasing with least upper bound equal to \sqrt{N} for $x \le \sqrt{N}$. Thus convergence is guaranteed for any positive starting value. Equation (6) shows that the convergence is of order k.

The first two estimates of $\sqrt{2}$ (1.414213562) obtained with

$$F_4(x) = (x^4 + 6x^2N + N^2)/(4x^3 + 4xN)$$
(9)

and starting value 1 are 17/12 = 1.4166666667 and 665857/470832 = 1.414213562, rounded to nine decimal places; convergence is rapid.

The Newton method is

$$F_2(x) = (x^2 + N)/2x.$$
 (10)

The approximation of Al-Kalsadi is

$$\sqrt{(a^2+b)} \approx (4a^3+3ab)/(4a^2+b),$$
 (11)

which is $F_3(a)$ with N replaced by $a^2 + b$; with starting value 1 (a = 1) the first estimate of $\sqrt{2}$ is 7/5 = 1.4.

NOTES

REFERENCES

- 1. J. Bentley, Programming pearls—birth of a cruncher, Communications of the Association for Computing Machinery 29 (1986) 1155-1161.
- 2. E. Frank, On continued fractions for binomial quadratic surds, Numerische Mathematik 4 (1962) 85–95.
- 3. F. Cajori, A history of mathematics, second edition, Macmillan, New York, 1958.
- 4. H. Davenport, *The higher arithmetic—an introduction to the theory of numbers*, Hutchinson's University Library, London, 1952.

Department of Computing Science, University of Glasgow mjj@dcs.glasgow.ac.uk

A Note on Jacobi Symbols and Continued Fractions

A. J. van der Poorten and P. G. Walsh

1. INTRODUCTION. It is well known that the continued fraction expansion of a real quadratic irrational is periodic. Here we relate the expansion for \sqrt{rs} , under the assumption that $rX^2 - sY^2 = \pm 1$ has a solution in integers X and Y, to that of $\sqrt{r/s}$ and to the Jacobi symbols $\left(\frac{r}{s}\right)$, which appear in the theory of quadratic residues.

We have endeavoured to make our remarks self-contained to the extent of providing a brief reminder of the background theory together with a cursory sketch of the proofs of the critical assertions. For extensive detail the reader can refer to [5], the bible of the subject. The introductory remarks following in Sections 2-3 below are *inter alia* detailed in [1].

Let p and q denote distinct odd primes. In [3], Friesen proved connections between the value of the Legendre symbol $\left(\frac{p}{q}\right)$ and the length of the period of the continued fraction expansion of \sqrt{pq} . These results, together with those of Schinzel in [6], provided a solution to a conjecture of Chowla and Chowla in [2].

We report a generalization of those results to the evaluation of Jacobi symbols $\left(\frac{r}{s}\right)$, and, in the context of there being a solution in integers X, Y to the equation $rX^2 - sY^2 = \pm 1$, remark on the continued fraction expansion of $\sqrt{r/s}$ vis à vis that of \sqrt{rs} .

Theorem 1. Let r and s be squarefree positive integers with r > s > 1, such that the equation $rX^2 - sY^2 = \pm 1$ has a solution in positive integers X, Y. Suppose the continued fraction expansion of \sqrt{rs} is $[a_0, \overline{a_1, a_2, \ldots, a_l}]$. Then both the length of the period l = 2h, and the 'central' partial quotient a_h , are even, and the continued fraction expansion of \sqrt{rs} is

$$\left[\frac{1}{2}a_h, \overline{a_{h+1}, \ldots, a_l, a_1, \ldots, a_h}\right] = \left[\frac{1}{2}a_h, \overline{a_{h-1}, \ldots, a_1, a_l, a_1, \ldots, a_{h-1}, a_h}\right].$$