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# The Area of the Medial Parallelogram of a Tetrahedron

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David N. Yetter

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The midpoints of any four edges of a Euclidean tetrahedron that form a cycle are coplanar, and are the vertices of a parallelogram. The purpose of this note is to derive a simple formula for the area of this *medial parallelogram* of a tetrahedron in terms of the lengths of the six edges. It would appear that this result is either new or long-forgotten.

Despite the very classical nature of the problem our formula solves, there is some serious contemporary interest arising from recently proposed simplicial models for quantum gravity, in which such a formula is needed to approach the problem of length operators; see [1], [2].

Consider a tetrahedron with edge-lengths as in Figure 1. Fix a pair of non-incident edges, say those of lengths  $e$  and  $f$ . It is then easy to see that the midpoints of the remaining four edges lie in a plane parallel to both of the chosen edges, and equidistant from the planes containing each chosen edge and parallel to both, and that they form the vertices of a parallelogram in this plane.

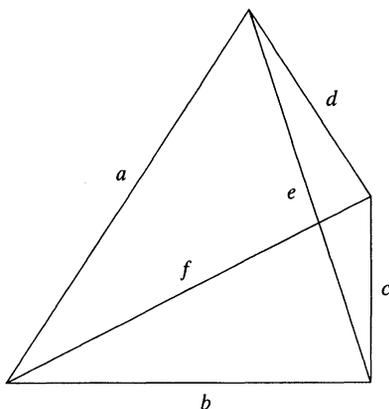


Figure 1. A generic tetrahedron

**Definition 1.** Given a pair of non-incident edges in a tetrahedron, the *medial parallelogram* determined by the pair is the parallelogram whose vertices are the midpoints of the remaining four edges.

Our main result is

**Theorem 2.** The area of the medial parallelogram determined by the edges of lengths  $e$  and  $f$  in the tetrahedron of Figure 1 is

$$\frac{1}{8} \sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}$$

*Proof:* The key is to consider the vertices of the tetrahedron as vectors  $2\vec{p}$ ,  $2\vec{q}$ ,  $2\vec{r}$ , and  $2\vec{s}$  in  $\mathbb{R}^3$ . The factors of 2 in the vertices given as vectors are included to avoid fractions; see Figure 2.

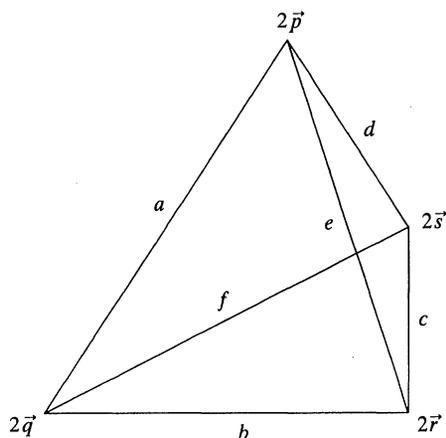


Figure 2. Tetrahedron with vertices as vectors

The vertices of the medial parallelogram are then given by the vectors  $\vec{p} + \vec{q}$ ,  $\vec{q} + \vec{r}$ ,  $\vec{r} + \vec{s}$ , and  $\vec{s} + \vec{p}$ . The lengths of the six edges are given in terms of the six vectors by

$$a = 2|\vec{p} - \vec{q}|, \quad b = 2|\vec{q} - \vec{r}|, \quad c = 2|\vec{r} - \vec{s}|,$$

$$d = 2|\vec{s} - \vec{p}|, \quad e = 2|\vec{r} - \vec{p}|, \quad f = 2|\vec{s} - \vec{q}|.$$

The medial tetrahedron is then spanned by the vectors  $\vec{u} = \vec{r} - \vec{p}$  and  $\vec{v} = \vec{s} - \vec{q}$ ; see Figure 3.

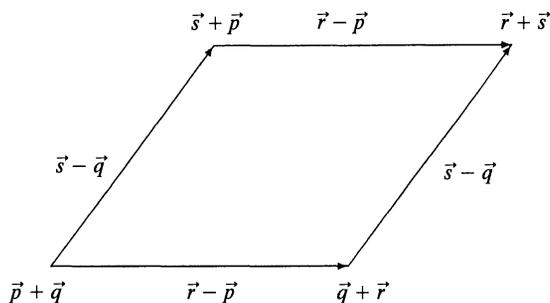


Figure 3. The medial tetrahedron in terms of vectors

The area of the medial tetrahedron is thus  $|\vec{u} \times \vec{v}|$ .

Now, recall that since  $\sin^2 \theta = 1 - \cos^2 \theta$ , the vector (cross) and scalar (dot) products of an two vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^3$  are related by

$$|\vec{x} \times \vec{y}|^2 = |\vec{x}|^2 |\vec{y}|^2 - (\vec{x} \cdot \vec{y})^2.$$

thus, in our case we have

$$\begin{aligned}
 \text{Area}^2 &= |\vec{u} \times \vec{v}|^2 \\
 &= |\vec{r} - \vec{p}|^2 |\vec{s} - \vec{q}|^2 - [(\vec{r} - \vec{p}) \cdot (\vec{s} - \vec{q})]^2 \\
 &= \frac{1}{16} e^2 f^2 - [\vec{r} \cdot \vec{s} - \vec{r} \cdot \vec{q} - \vec{p} \cdot \vec{s} + \vec{p} \cdot \vec{q}] \\
 &= \frac{1}{16} e^2 f^2 - \frac{1}{4} [-2\vec{r} \cdot \vec{s} - (-2)\vec{r} \cdot \vec{q} - (-2)\vec{p} \cdot \vec{s} + (-2)\vec{p} \cdot \vec{q}]^2 \\
 &= \frac{1}{16} e^2 f^2 - \frac{1}{4} [ (|\vec{r}|^2 - 2\vec{r} \cdot \vec{s} + |\vec{s}|^2) - (|\vec{r}|^2 - 2\vec{r} \cdot \vec{q} + |\vec{q}|^2) \\
 &\quad - (|\vec{p}|^2 - 2\vec{p} \cdot \vec{s} + |\vec{s}|^2) + (|\vec{p}|^2 - 2\vec{p} \cdot \vec{q} + |\vec{q}|^2) ]^2 \\
 &= \frac{1}{16} e^2 f^2 - \frac{1}{4} [ |\vec{r} - \vec{s}|^2 - |\vec{r} - \vec{q}|^2 - |\vec{p} - \vec{s}|^2 + |\vec{p} - \vec{q}|^2 ]^2 \\
 &= \frac{1}{16} e^2 f^2 - \frac{1}{4} \left[ \left(\frac{c}{2}\right)^2 - \left(\frac{b}{2}\right)^2 - \left(\frac{d}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \right]^2 \\
 &= \frac{1}{64} [4e^2 f^2 - (a^2 - b^2 + c^2 - d^2)^2]
 \end{aligned}$$

Thus, taking square roots, we have the desired result. ■

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