The Area of the Medial Parallelogram of a Tetrahedron

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The midpoints of any four edges of a Euclidean tetrahedron that form a cycle are coplanar, and are the vertices of a parallelogram. The purpose of this note is to derive a simple formula for the area of this *medial parallelogram* of a tetrahedron in terms of the lengths of the six edges. It would appear that this result is either new or long-forgotten.

Despite the very classical nature of the problem our formula solves, there is some serious contemporary interest arising from recently proposed simplicial models for quantum gravity, in which such a formula is needed to approach the problem of length operators; see **[1]**, **[2]**.

Consider a tetrahedron with edge-lengths as in Figure 1. Fix a pair of non-incident edges, say those of lengths e and f. It is then easy to see that the midpoints of the remaining four edges lie in a plane parallel to both of the chosen edges, and equidistant from the planes containing each chosen edge and parallel to both, and that they form the vertices of a parallelogram in this plane.

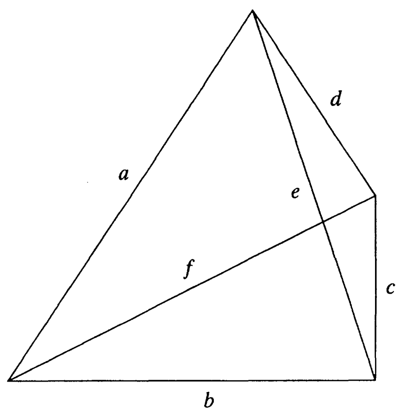

Figure 1. A generic tetrahedron

Definition 1. Given a pair of non-incident edges in a tetrahedron, the *medial parallelogram* determined by the pair is the parallelogram whose vertices are the midpoints of the remaining four edges.

Our main result is

Theorem 2. *The area of the medial parallelogram determined by the edges of lengths e and f in the tetrahedron of Figure 1 is*

$$\frac{1}{8}\sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}$$

Proof: The key is to consider the vertices of the tetrahedron as vectors $2\vec{p}$, $2\vec{q}$, $2\vec{r}$, and $2\vec{s}$ in $\mathbb{R}^3$. The factors of 2 in the vertices given as vectors are included to avoid fractions; see Figure 2.

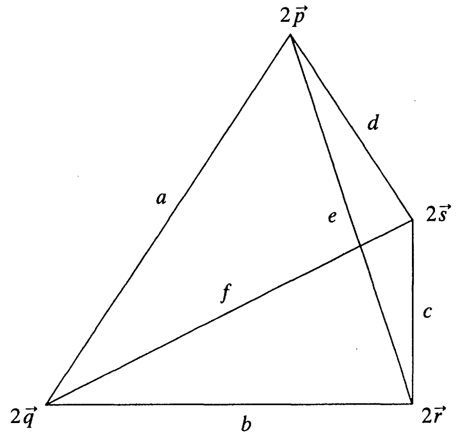

Figure 2. Tetrahedron with vertices as vectors

The vertices of the medial parallelogram are then given by the vectors $\vec{p}+\vec{q}$, $\vec{q}+\vec{r}$, $\vec{r}+\vec{s}$, and $\vec{s}+\vec{p}$. The lengths of the six edges are given in terms of the six vectors by

$$a = 2|\vec{p}-\vec{q}|, \quad b = 2|\vec{q}-\vec{r}|, \quad c = 2|\vec{r}-\vec{s}|,$$
$$d = 2|\vec{s}-\vec{p}|, \quad e = 2|\vec{r}-\vec{p}|, \quad f = 2|\vec{s}-\vec{q}|.$$

The medial tetrahedon is then spanned by the vectors $\vec{u}=\vec{r}-\vec{p}$ and $\vec{v}=\vec{s}-\vec{q}$; see Figure 3.

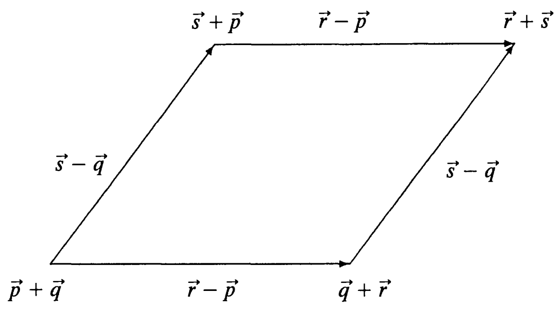

Figure 3. The medial tetrahedron in terms of vectors

The area of the medial tetrahedron is thus $|\vec{u}\times\vec{v}|$.

Now, recall that since $\sin^2\theta = 1-\cos^2\theta$, the vector (cross) and scalar (dot) products of an two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$ are related by

$$|\vec{x}\times\vec{y}|^2 = |\vec{x}|^2|\vec{y}|^2 - (\vec{x}\cdot\vec{y})^2.$$

thus, in our case we have

$$
\begin{aligned}
\text{Area}^2 &= |\vec{u} \times \vec{v}|^2 \\
&= |\vec{r}-\vec{p}|^2|\vec{s}-\vec{q}|^2 - \left[(\vec{r}-\vec{p})\cdot(\vec{s}-\vec{q})\right]^2 \\
&= \frac{1}{16}e^2f^2 - \left[\vec{r}\cdot\vec{s} - \vec{r}\cdot\vec{q} - \vec{p}\cdot\vec{s} + \vec{p}\cdot\vec{q}\right] \\
&= \frac{1}{16}e^2f^2 - \frac{1}{4}\left[-2\vec{r}\cdot\vec{s} - (-2)\vec{r}\cdot\vec{q} - (-2)\vec{p}\cdot\vec{s} + (-2)\vec{p}\cdot\vec{q}\right]^2 \\
&= \frac{1}{16}e^2f^2 - \frac{1}{4}\Big[\left(|\vec{r}|^2 - 2\vec{r}\cdot\vec{s} + |\vec{s}|^2\right) - \left(|\vec{r}|^2 - 2\vec{r}\cdot\vec{q} + |\vec{q}|^2\right) \\
&\qquad\qquad - \left(|\vec{p}|^2 - 2\vec{p}\cdot\vec{s} + |\vec{s}|^2\right) + \left(|\vec{p}|^2 - 2\vec{p}\cdot\vec{q} + |\vec{q}|^2\right)\Big]^2 \\
&= \frac{1}{16}e^2f^2 - \frac{1}{4}\left[|\vec{r}-\vec{s}|^2 - |\vec{r}-\vec{q}|^2 - |\vec{p}-\vec{s}|^2 + |\vec{p}-\vec{q}|^2\right]^2 \\
&= \frac{1}{16}e^2f^2 - \frac{1}{4}\left[\left(\frac{c}{2}\right)^2 - \left(\frac{b}{2}\right)^2 - \left(\frac{d}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]^2 \\
&= \frac{1}{64}\left[4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2\right]
\end{aligned}
$$

Thus, taking square roots, we have the desired result. ■

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