Now let $f_1, \dots, f_n$ be continuous functions that are orthonormal in H. For all real numbers $a_1, \dots, a_n$ and all $x \in [0, 1]$, we have

$$\sum_{i=1}^{n} a_i f_i(x) \le K \cdot \left\| \sum_{i=1}^{n} \alpha_i f_i \right\|_2 = K \sqrt{\sum_{i=1}^{n} \alpha_i^2}.$$

Fix $x \in [0, 1]$, and let $\alpha_i = f_i(x)$. Then $\sum_{i=1}^{n} (f_i(x))^2 \le K \cdot \sqrt{\sum_{i=1}^{n} (f_i(x))^2}$, so $\sum_{i=1}^{n} (f_i(x))^2 \le K^2$. Integrating both sides from 0 to 1 gives $n \le K^2$. Thus every orthonormal set of continuous functions in H has at most K^2 elements. This contradicts the assumption that H is infinite-dimensional.

The conclusion does not follow with $(0, 1]$ in place of $[0, 1]$. For $n = 1, 2, \dots$, let $f_n\colon [0, 1] \to \mathbf{R}$ be a continuous function with $\|f_n\|_2 = 1$ and support in $(1/(n+1), 1/n)$. Then $\{f_n\}$ is an orthonormal set, so the map $\Phi\colon l^2 \to L^2[0, 1]$ given by $\Phi(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n$ is a linear isometry. In addition, each $\Phi(\alpha)$ is continuous on $(0, 1]$, since for all $x \in (0, 1]$ there exists an open interval I about x such that $f_n \ne 0$ on I for at most one n. Thus the range of Φ is a closed, infinite-dimensional subspace of $L^2(0, 1]$ whose elements are continuous functions.

The first part of this problem is contained in problems 28 and 55 in Chapter 10 of H. L. Royden, *Real Analysis*, Third Edition, Macmillan, 1988. The solution here follows Royden's generous hints.

Solved also by P. J. Fitzsimmons, P. M. Jarvis, J. H. Lindsey II, A. Sasane (The Netherlands), and the proposers.

Two Recurrence Relations, One Easy, One Hard

10670 [1998, 559]. *Proposed by Salomon Benchimol and Elliott Cohen, Paris, France.*
(a) For which values of $u_0 > 0$ and $u_1 > 0$ does the sequence defined by $u_{n+2} = 1 + u_{n+1}/u_n$ for $n \ge 0$ converge?
(b) For which values of $u_0 > 0$ and $u_1 > 0$ does the sequence defined by $u_{n+2} = 1 + u_n/u_{n+1}$ for $n \ge 0$ converge?

Solution of part **(a)** *by Con Amore Problems Group, Copenhagen, Denmark.* This sequence converges to 2 for every choice of $u_0, u_1 > 0$. Clearly $u_n > 0$ for all n, so $u_n = 1 + u_{n-1}/u_{n-2} > 1$ for $n \ge 2$. If $n \ge 5$, then $u_n = 1 + u_{n-1}/u_{n-2} = 1 + 1/u_{n-3} + 1/u_{n-2} < 3$. This proves the $k = 0$ case of the following claim: For any $k \ge 0$,

$$u_n > \frac{2^{2k+2} - 1}{2^{2k+1} + 1} \text{ for } n \ge 6k + 2, \qquad \text{and} \qquad u_n < \frac{2^{2k+3} + 1}{2^{2k+2} - 1} \text{ for } n \ge 6k + 5.$$

This proves convergence, since both of these bounds converge to 2 as $k \to \infty$. We prove the claim by induction. Choose $k \ge 1$ and assume that the claim holds for smaller values of k. For $n \ge 6(k-1) + 5 = 6k - 1$, we have

$$u_n < \frac{2^{2(k-1)+3} + 1}{2^{2(k-1)+2} - 1} = \frac{2^{2k+1} + 1}{2^{2k} - 1}.$$

Therefore, for $n \ge 6k + 2$, we have

$$u_n = 1 + \frac{1}{u_{n-2}} + \frac{1}{u_{n-3}} > 1 + 2\,\frac{2^{2k} - 1}{2^{2k+1} + 1} = \frac{2^{2k+2} - 1}{2^{2k+1} + 1},$$

as required. For $n \ge 6k + 5$, we then have

$$u_n = 1 + \frac{1}{u_{n-2}} + \frac{1}{u_{n-3}} < 1 + 2\,\frac{2^{2k+1} + 1}{2^{2k+2} - 1} = \frac{2^{2k+3} + 1}{2^{2k+2} - 1},$$

as required.

Editorial comment. No correct solutions of **(b)** were received. It appears that the set of pairs (x, y) such that the sequence defined by $u_0 = x, u_1 = y, u_{n+2} = 1 + u_n/u_{n+1}$ converges is a curve through (2, 2) of the form

$$y = 2 + \frac{1}{2}(x-2) - \frac{1}{20}(x-2)^2 + \frac{7}{600}(x-2)^3 - \frac{71}{20400}(x-2)^4 + \cdots.$$

Part **(a)** solved also by S. S. Kim and the proposer.

The Number of Zeros of a Maclaurin Polynomial

10671 [1998, 559]. *Proposed by F. Rothe, University of North Carolina, Charlotte, NC.* Let

$$P_n(x) = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

be the Maclaurin polynomial of order $2n+1$ of the sine function. Let c_n be the number of real zeros of P_n. Determine $\lim_{n\to\infty} c_n/(2n+1)$.

Composite solution by Sung Soo Kim, Hanyang University, Ansan, Kyunggi, Korea, and the editors. The integral form of Taylor's theorem tells us that

$$P_n(x) = \sin x + \frac{(-1)^n}{(2n+1)!}e_{2n+1}(x), \qquad \text{where} \qquad e_k(x) = \int_0^x (x-t)^k \sin t\, dt.$$

Now $e_1(x) = x - \sin x$ and is positive for all $x > 0$, and $e_k'(x) = ke_{k-1}(x)$ for $k > 1$. Thus for $k \geq 3$, $e_k(x)$ is positive, increasing, and convex (concave up) on $(0, \infty)$.

Let $f_n(x) = e_{2n+1}(x)/(2n+1)!$. We now consider the intervals $[a, b]$ on which $\sin x$ is monotone. Suppose first that n is even and $f_n(b) < 1$. If $a = (2m - 1/2)\pi$ and $b = (2m + 1/2)\pi$, then $P_n(x)$ is negative at a and positive at b and strictly increasing on $[a, b]$ so there is exactly one zero of P_n in $[a, b]$. If instead $a = (2m + 1/2)\pi$ and $b = (2m + 3/2)\pi$, then $P_n(x)$ is positive at a and negative at b. If $c = (2m+1)\pi$, then P_n is positive on $[a, c]$. Thus P_n has at least one real zero in $[c, b]$. If there were more than one zero in $[c, b]$, there would have to be some $z \in [c, b]$ with $P_n''(z) < 0$: a convex function cannot be zero at more than one point on an interval if it is positive at one end and negative at the other. But $\sin(x)'' > 0$ on $[c, b]$, so also $P_n'' > 0$ on $[c, b]$, which is a contradiction. The case where n is odd is similar. The final case to be considered is when $f_n(a) < 1 < f_n(b)$. Here there can be two zeros in the interval, but again considerations of convexity forbid more.

This shows that the number of real zeros of P_n differs by at most a constant from the number of intervals $(k - \pi/2, k + \pi/2)$ in which $f_n < 1$. That number is given to within a bounded error by $2B(n)/\pi$, where $B(n)$ is the unique positive solution to $f_n(x) = 1$. But

$$e_k(x) = \int_0^x (x-t)^k \sin t\, dt < \int_0^\pi (x-t)^k \sin t\, dt \lesssim \pi x^k,$$

while

$$e_k(x) > \int_0^{2\pi} (x-t)^k \sin t\, dt = \sum_{j=0}^{k}\binom{k}{j}(x-\pi)^{k-j}\int_{-\pi}^{\pi} u^j \sin u\, du > 2\pi k(x-\pi)^{k-1}.$$

Thus $B(n)$ lies between the solutions to $x^{2n+1} = (2n+1)!/\pi$ and $(x-\pi)^{2n} = (2n)!/(2\pi)$. Both are asymptotically $2n/e$ by Stirling's formula, so $B(n) \approx 2n/e$. Thus, the number c_n of real zeros of $P_n(x)$ is asymptotic to $4n/e\pi$, so that $c_n/(2n+1) \approx 2/e\pi$.

Editorial comment. David Bradley pointed out that the result is known and may be found (with details for the cosine function) in G. Szegő, Über eine Eigenschaft der Exponentialreihe, in *Gábor Szegő: Collected Papers 1915–1927*, Birkhauser, 1982, p. 659).

Solved also by J. H. Lindsey II, GCHQ Problems Group, and the proposer.